

**BLOCK UNIFICATION AND BOUNDARY VALUE NUMERICAL
INTEGRATION SCHEMES FOR ORDINARY AND PARTIAL
DIFFERENTIAL EQUATIONS**

MODEBEI, MARK IFEANYI
(15/68DR003)

FEBRUARY, 2020.

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DIFFERENTIAL EQUATIONS**

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BEING A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS,
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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF
DOCTOR OF PHILOSOPHY (Ph.D.) DEGREE IN MATHEMATICS.

FEBRUARY, 2020

CERTIFICATION

This is to certify that the research work reported in this thesis was done by MODEBEI, Mark Ifeanyi with Matriculation number 15/68DR003 in the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for the award of a Doctor of Philosophy Degree in Mathematics.

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DEDICATION

This research work is dedicated to the Almighty God, the author and giver of wisdom.

DECLARATION

I, Mark Ifeanyi MODEBEI, a Ph.D. student in the Department of Mathematics, University of Ilorin, Ilorin, hereby declare that this thesis entitled "Block Unification and Boundary Value Numerical Integration Scheme for Ordinary and Partial Differential Equations", submitted by me is based on my actual and original work. Any materials obtained from other sources or work done by any other persons or institutions have been duly acknowledged. In addition, the research has been approved by the University of Ilorin Ethical Review Committee.

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ABSTRACT

Many physical problems can be modelled as Differential Equations (DEs). DEs require solutions, either in closed form (analytic form) or in approximate form. Closed form solutions are rare to obtain for some DEs of high order. Hence, the need for numerical methods/techniques. The difficulties in obtaining closed form solutions for higher order Boundary Value Problems (BVPs) in Ordinary Differential Equation (ODEs) and Initial-Boundary Value Problems (IBVPs) in Partial Differential Equations (PDEs) are the motivations for this study. This study was aimed at developing new classes of continuous block integrators for solution of fourth and fifth order BVPs and IBVPs. The objectives were to: (i) develop new classes of continuous block integrators for step number $k = 5$; (ii) analyse the developed integrators for consistency and convergence; (iii) apply the derived block integrators to solve the IBVPs; and (iv) compare the efficiency of the new methods with some existing methods in literature.

The fourth and fifth order ODEs of the form:

$$y^{(m)} = f(x, y, y', \dots, y^{(m-1)}), \quad a \leq x \leq b \quad (1)$$

where $m = 4, 5$ with appropriate boundary conditions were considered. The assumed trial solution is of the form

$$p(x) = \sum_{i=0}^k \rho_i T_i^*(x) \cong y(x) \quad (2)$$

where ρ_i 's are real constants, $y \in C^m(a, b)$, and $T_i^*(x)$ is the i^{th} degree shifted Chebyshev polynomial of the first kind over the interval $[0, k]$. In order to determine the unknown coefficients ρ_i 's of (2) the process of collocation was adopted. This yielded the desired

continuous block integrators. In order to extend the scope of application of the methods to IBVPs, the Method of Lines was adopted to semi-discretize PDEs

The findings of this study were:

- (i) two new classes of continuous implicit five-step methods namely: Block Unification Method (BUM) and Boundary Value Method (BVM) for direct solution of general fourth and fifth order BVPs were derived;
- (ii) the new classes of methods were proved to be consistent and convergent;
- (iii) new class of continuous implicit five-step methods were extended to solve fourth and fifth order IBVPs;
- (iv) the efficiency of the BUM and BVM compare favourably with some existing methods in literature in terms of accuracy;
- (v) the BUMs and BVMs are self starting, and admit easy change of step-size and function evaluation at off-step points.

The study concluded that the proposed two new classes of methods are efficient, and when compared with some existing methods in literature are accurate. The proposed methods are therefore recommended for the solution of fourth and fifth order BVPs and IBVPs.

CHAPTER ONE: GENERAL INTRODUCTION

1.1 Background of the Study

In physical sciences, problems are translated to mathematical language and then into mathematical models. These models aid understanding of the phenomenon. In all cases, these models involve equations and in most cases too, these equations result into differential equations which could be ordinary, partial, delay, fractional and stochastic differential equations to mention but few.

Differential equations find its applications in sciences, engineering, medicine and social science including, Economics, Operation Research. such application are but not limited to motions of objects and the planet at large as seen in Newton's laws of motion, Kepler's law of planetary motions and gravitation force, Hooke's law of elasticity, the simple pendulum, Ohm's law in electrical circuits, problems in chemical kinetics, marginal cost and values in economics, population growth (of people, bacterial, molecules etc.) in mathematical biology, Cantilever beam construction and bridges in engineering, magnetohydrodynamic (MHD) flow of fluid in fluid mechanics and a host of others. In all these, the models often results into equations that contain some derivatives of unknown dependent function of one or several independent variables. If the model have derivatives of unknown dependent variables of one independent variable, then such model is termed an Ordinary Differential Equation (ODE). If the model have derivatives of unknown dependent variables of two or several independent variables, then such model is termed a Partial Differential Equation (PDE). This work concerns the computational study of the numerical solutions of Ordinary and Partial differential equations.

The developed differential equations require solutions, either in closed form (analytic form) or in numerical forms. In most cases, closed form of solutions are rare to come by as there exists limited methods for solving such models in the form of differential equations. This brings to light the use of numerical methods/techniques to solve the modeled problem.

Numerical techniques are numerous and the types know no bounds. They include: Euler method, Runge-Kutta methods, Linear multistep method, Shooting method, Finite difference method, Finite element methods. Other methods are: Spectral method based on the Fourier transformation, Method of lines which reduces the PDE to a large system of ordinary differential equations, Boundary Element Method (BEM) based on transforming a PDE to an integral equation on the boundary of the domain, Finite Volume Method based on dividing the domain into many small domains and it is popular in computational fluid dynamics, the list is endless.

1.2 Aim and Objectives

The aim of this study is to construct linear multistep methods to solve higher order differential equations with specified boundary conditions, and to achieve this aim, the objectives were:

- (i) use the Chebyshev polynomials as basis to develop a class of continuous implicit k-step linear multistep methods by interpolation and collocation approach;
- (ii) construct block methods of discrete linear multistep methods;
- (iii) use the method of lines to reduce Partial differential equations to system of Ordinary differential equations (i);
- (iv) analyze the developed methods for its order, local truncation error, consistency and convergence; and
- (v) make comparison of the developed methods in terms of accuracy, flexibility in terms of implementation and efficiency in terms of CPU time, with existing methods in literature.

1.3 Statement of the Problem

Many differential equations (ordinary and partial) do not have analytic solutions and most numerical methods for the solutions of such differential equations involve reducing higher order differential equations into system of first order equations. This no doubt involves many subroutines in programming and thereby increases computational cost. Furthermore, Linear Multistep Methods (LMMs) are extensively used for the numerical approximation of Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs). However, they are not commonly used to approximate continuous Boundary Value Problems (BVPs), except in connection with the other methods that are continuous like the shooting method. Recently, Block unification and Boundary Value Methods (BVMs) have been considered for the approximation of Initial-Boundary Value Problems (IBVPs) Biala, Jator and Adeniyi (2016). It would be shown that these methods can also be conveniently used to approximate, with high accuracy, continuous IBVPs of both the ordinary and partial differential equations. These methods have several advantages among which are the fact that they do not require any starting value to kick-start their implementation, they are self-starting. They also have the advantage of producing smaller global errors (at the end of the range of integration) than those produced by the step-by-step methods due to the presence of accumulated errors at each step in the step-by-step method. The development of Block Unification Method (BUM) and Boundary Value Method (BVM) for solving IBVPs in both ordinary and partial differential equations is the main concern of this study.

1.4 Justification for the Study

Fourth and fifth order BVPs in ODEs and IBVPs in PDEs have numerous applications in Engineering and Sciences. The need for solutions of these classes of problems is on the increase. Available methods for their solution involve cumbersome computations which are invariably time consuming and the maximum errors can be quite large (see Modebei, Otaide, Olaiya and Ejiofor (2015), Mohanty and Kaur (2017), Pandey (2017)). This re-

search is aimed at addressing these setbacks. The proposed methods are less expensive in terms of the number of function evaluations per step, highly efficient in terms of accuracy; easy to automate resulting in economy of computer time, and is also flexible in change of step size.

1.5 Scope of the Study

Two new families of Linear Multistep Methods (LMMs) are derived by method of interpolation and collocation, using the shifted Chebyshev polynomials as basis functions in the trial solution. These methods are discussed as boundary value methods and block unification methods for the numerical approximation of the general fourth and fifth order initial-boundary value problems (IBVPs) in ordinary and partial differential equations. The convergence of these families of methods is also established.

1.6 Definition of Terms

Definition 1.6.1 (Initial Value Problems). *An n^{th} order scalar initial value problem in ordinary differential equations consists of two parts: An ordinary differential equation which can be written in the form*

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \quad (1.6.1)$$

and the initial conditions

$$y(a) = a_0, y'(a) = a_1, \dots, y^{(n-1)}(a) = a_{n-1} \quad (1.6.2)$$

where f is continuous in some open set $\Omega = (a, b)$ in the (x, y) plane, $a, b, a_i, i = 0, 1, \dots, n$ are real constants. The order of an ODE is the order of the highest-order derivative in the differential equation.

Definition 1.6.2 (Boundary Value Problems). *A boundary value problem is a problem,*

usually an ordinary differential equation or a partial differential equation, which has values assigned on the physical boundary of the domain in which the problem is specified. For example,

$$\begin{aligned} \frac{\partial^{(n)}u}{\partial x^n} - \Delta^n u &= f \text{ in } \Omega \\ u(0, x) &= u_1 \text{ on } \partial\Omega; \\ \frac{\partial u}{\partial x}(0, x) &= u_2 \text{ on } \partial\Omega \end{aligned} \tag{1.6.3}$$

where Ω is the domain and $\partial\Omega$ is its boundary.

Definition 1.6.3 (Collocation). This is a projection approach for solving differential equations in which the approximate solution is determined from the condition that the equation be satisfied at some chosen given points. The points at which the differential equation is satisfied are called collocation points.

Definition 1.6.4 (Interpolation). This is the approach where a trial solution of an equation is evaluated at some selected points. These selected points are called interpolation points.

Definition 1.6.5 (Step-size). Given the sequence of points $\{x_i\}$ in the interval $I = [a, b]$ defined by $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ such that $h = x_i - x_{i-1}$, $i = 0, 1, 2, \dots, N$. The parameter h is called the mesh size or step-size while N is called the number of subintervals.

Definition 1.6.6 (One-Step Method). Solutions of ordinary differential equations can be derived numerically using the one step methods and the Multistep Methods. The one-step methods use data at a single point x_n , say, to predict the solution at point x_{n+1} . For an IVP, a one-step method is described as;

$$y_{n+1} = y_n + hf_n \tag{1.6.4}$$

where h is the step-size, $y_n \in y(x_n)$, $f_n \in f(x_n, y_n)$.

Remark 1.6.1. The method in (1.6.4) is termed an explicit form while an implicit form is defined thus;

$$y_{n+1} = y_n + hf(x_n, y_n, y_{n+1}) \quad (1.6.5)$$

This is because, unlike (1.6.4), the data x_{n+1} appears also on the right hand side of (1.6.5). Examples of one-step methods include: the backward and forward Euler's methods, the midpoint method, the trapezoid method and the Runge-Kutta methods.

On the other hand, LMMs require that two or more past values are known in order to determine the value of y_{n+1} .

Definition 1.6.7 (k-step Method). The most general k-step method takes the form

$$\sum_{i=0}^k \alpha_i y_{n+i} = h^\mu \sum_{i=0}^k \beta_i f_{n+i} \quad (1.6.6)$$

where

$y_{n+i} \approx y(x_{n+i}) \approx y(x_n + ih)$, $f_{n+i} = f(x_{n+i}, y_{n+i}) = f(x_n + ih, y(x_n + ih))$, and is of implicit type unless $\beta_k = 0$, when it becomes explicit. The coefficients have been normalized so that the coefficient of y_{n+k} is $\alpha_k = 1$; μ is the order of the differential equation.

Remark 1.6.2. The k-step LMM (1.6.6) is called linear because it involves only linear combinations of the y_{n+k} and the f_{n+k} .

Definition 1.6.8 (Characteristic polynomials of k-step method). The k-step method has first and second characteristic polynomials

$$\begin{aligned} \rho(r) &= r^k + \alpha_{k-1}r^{k-1} + \dots + \alpha_0, \\ \sigma(r) &= \beta_k r^k + \beta_{k-1}r^{k-1} + \dots + \beta_0 \end{aligned} \quad (1.6.7)$$

Definition 1.6.9. (Linear difference operator) The linear difference operator L_h associated with the LMM (1.6.6) is defined for an arbitrary continuously differentiable function

$z(x)$ by

$$\mathbf{L}_h[z(x)] \equiv h \sum_{j=0}^{m(m)} y^{(m)}(x+jh) - \sum_{j=0}^{k-1} a_j y(x+jh) - h \sum_{j=0}^{\mu} \beta_j y^{(\mu)}(x+jh). \quad (1.6.8)$$

where $m = 0(1)k - 1$, $y^{(\mu)} = f$, and μ is the order of the differential equation.

Expanding (1.6.8) in Taylor series about the point $h = 0$, the following is obtained

$$\mathbf{L}_h[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + O(h^{p+1}) \quad (1.6.9)$$

where $C_0, C_1, C_2, \dots$ are linear combinations of the coefficients $a_0, a_1, \dots, a_k$ ($a_k = 1$), $\beta_0, \beta_1, \dots, \beta_k$.

Definition 1.6.10. (Order and Local truncation error (LTE)) The LMM (1.6.6) is of order p if $C_0 = C_1 = C_2 = \cdots = C_{p+\mu} = 0$, and $C_{p+\mu+1} \neq 0$ in which

$$\mathbf{L}_h[y(x); h] = C_{p+\mu+1} h^{p+\mu+1} y^{(p+\mu+1)}(x) + O(h^{p+\mu+2}) \quad (1.6.10)$$

In this case, $C_{p+\mu+1}$ is the local truncation error or simply principal error constant.

Definition 1.6.11. (Consistency) A linear difference operator $\mathbf{L}_h$ is said to be consistent of order p if it satisfies (1.6.10) with $p > 0$ for every smooth function y .

Remark 1.6.3. An LMM whose difference operator is consistent of order p for some $p > 0$ is said to be consistent.

Definition 1.6.12 (Root Condition). A polynomial is said to satisfy the root condition if all its roots lie within or on the boundary of a unit circle, with those on the boundary being simple. In other words, all roots satisfy $|r| < 1$ and any that satisfy $|r| = 1$ are simple.

Remark 1.6.4. We say that λ is a simple root of $\rho(r)$ if $(\lambda - r)$ is a factor of $\rho(r)$.

Definition 1.6.13 (Zero Stability). An LMM is said to be zero-stable if its first characteristic polynomial $\rho(r)$ satisfies the root condition.

Definition 1.6.14 (Convergence). *The LMM (1.6.6) is said to be convergent if, for all IVPs that possess a unique solution $y(x)$ for $x \in [x_0, x_N]$,*

$$\lim_{h \rightarrow 0} y_n = y_{x^*}, \quad nh = x^* - x_0 \quad (1.6.11)$$

holds for all $x^* \in [x_0, x_N]$

Henrichi, (1962); Lambert, (1967).

1.7 Chebyshev Polynomials

Definition 1.7.1 (Chebyshev polynomials of the first kind). *Chebyshev polynomials defined by $T_n(x)$ of the **first kind** defined on the interval $[-1, 1]$ is given by*

$$T_n(x) = \cos(n \cos^{-1} x), \quad n \geq 0 \quad (1.7.1)$$

and satisfies the recurrence relation

$$T_0(x) = 1; \quad T_1(x) = x; \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x); \quad n \geq 1 \quad (1.7.2)$$

In particular, $T_n(x)$ is indeed an algebraic polynomial of degree n with leading coefficient 2^{n-1} .

1.7.1 Properties of the Chebyshev polynomials $T_n(x)$

(a) **Derivatives:**

$$\begin{aligned} T_0(x) &= T_1'(x), \\ T_1(x) &= \frac{1}{4} T_2'(x), \\ T_n(x) &= \frac{1}{2} \left[\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1} \right], \quad n \geq 2, \quad x \in [-1, 1] \end{aligned} \quad (1.7.3)$$

$$(1 - x^2)T_n'(x) = n[xT_n(x) - T_{n+1}(x)] = n[T_{n-1}(x) - xT_n(x)], \quad x \in [-1, 1] \quad (1.7.4)$$

(b) **Multiplication:**

$$2T_r(x)T_q(x) = T_{r+q}(x) + T_{r-q}(x), \quad x \in [-1, 1] \quad (1.7.5)$$

(c) **Orthogonality:**

$$\int_{-1}^1 T_r(x)T_q(x)w(x)dx = \begin{cases} 0 & : r \neq q \\ \pi/2 & : r = q \neq 0 \\ \pi & : r = q = 0 \end{cases} \quad (1.7.6)$$

where $w(x) = (1 - x^2)^{-1/2}$ is the weight function in the interval $(-1, 1)$

(d) **Discrete orthogonality:**

With the zeros of $T_{n+1}(x)$ as node, $n > 0$, $r, q \leq n$ and $x_j = \arccos(\frac{(2n-1)\pi}{2n})$, Then

$$\sum_{j=0}^n T_r(x_j)T_q(x_j) = \begin{cases} 0 & : r \neq q \\ n/2 & : r = q \neq 0 \\ n & : r = q = 0 \end{cases} \quad (1.7.7)$$

for $1 \leq r \leq n$.

Definition 1.7.2 (Shifted Chebyshev polynomial). *The shifted Chebyshev polynomial on the interval $[a, b]$, is given as*

$$T_n^*(x) = T_n(y), \quad y = \frac{2}{b-a}x - \frac{a+b}{2} \quad (1.7.8)$$

with leading coefficient $2^{n-1} \frac{n!}{(b-a)^n}$.

Amparo, Javier and Nico (2007).

This is important to the development of the proposed LMMs, since the solution is sought in the interval $[x_{n+a}, x_{n+b}]$.

In this study, an approximate solution of an ordinary differential equation of the form

$$y(x) = \sum_{i=0}^k a_i T_i^*(x), \quad a \leq x \leq b \quad (1.7.9)$$

is sought, where $T_i^*(x)$ denotes the shifted Chebyshev polynomials of the first kind of degree i , a_i are unknown coefficients to be determined and k is the step length.

1.8 Organization of the Thesis

The thesis is organized as follow:

Chapter 1 is concerned with the introductory aspects comprising of the background, objectives, statement, justification, definition and organization of the thesis.

Chapter 2 deals with review of related literature.

Chapter 3 focuses on the methodology which includes the development and analysis of different schemes for BVPs in ODEs and IBVPs in PDEs.

Chapter 4 concerns the application of the derived schemes, results and discussion of the results.

Chapter 5 concludes the work with some remarks.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

Higher order ordinary differential equations of the form

$$y^{(\mu)} = f(x, y, y', \dots, y^{(\mu-1)}) \quad (2.1.1)$$

with appropriate boundary conditions find some of their applications in static deflection of a uniform beam. The deflection curve of each set of boundary conditions is such that beam with both ends embedded gives rise to Neumann and Dirichlet (first order) boundary conditions. A simply supported beam gives rise to Neumann and Dirichlet (second order) boundary conditions and cantilever beam left-end embedded, right-end free gives rise to a different condition. (Krajcinovic (1972) and Tseng (2014)). For other notable applications of ODEs, see Devakar, Sreenivasu and Shankar (2014).

Higher order boundary value problems occur, for example, in the modeling of induction motors with two rotor circuits. The behavior of this kind of circuits is governed by a fifth-order differential equation, (Richards and Sarma, 1994). Other applications of this type of boundary value problems arise in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences.

The fourth order PDE of the form;

$$y_{xxxx} = f(x, t, y_x, y_t, y_{xx}, y_{tt}) \quad (2.1.2)$$

and the fifth order PDE of the form;

$$y_{xxxxx} = g(x, t, y, y^2, y_x, y_t, y_{xx}, y_{tt}, y_{xxx}) \quad (2.1.3)$$

with appropriate initial-boundary conditions defined on the domain $\Omega = \{(x, t) : a <$

$x < b, c < t < d\}$, with the boundary $\partial\Omega$ is considered

The subscript notation denotes partial derivatives, $y_t = \partial y / \partial t$; $y_x = \partial y / \partial x$, and so on. For PDEs, a wide range of numerical methods have been used to solve higher order PDEs with Dirichlet and Neumann boundary conditions that arise in Engineering, Physics and Mathematics. Models formulated into PDEs of the type in (2.1.2) and (2.1.3) arise in sciences, for example, fourth and fifth-order Korteweg-de Vries (KdV) type equations, which arise naturally in modeling many different wave phenomena such as gravity-capillary waves, the propagation of shallow water waves over a flat surface and magneto-sound propagation in plasmas, (Kazeminia (2010); Goswami, Singh and Kumar (2016); Kumar *et al.*, 2015). Examples are a fully nonlinear Korteweg-de Vries (KdV)-like equation, the "good" Boussinesq equation, the Kaup-Kupershmidt equation and an extended KdV5 equation Goswami, Jagdev and Devendra (2017). These equations are used to model nonlinear dispersive waves in a wide range of application areas, such as water wave models, laser optics, and plasma physics, reaction diffusion systems, long waves on the interface between two viscous fluids (Hooper and Grimshaw, 1985) and thin hydrodynamics films, Sivashinsky (1983). Generally, these are PDEs with higher order spatial derivatives, Kakutani and Ono, (1969); Saucez, Vande, Schiesser and Zegeling (2014); Goswami *et al.*, (2017).

2.2 Review of Literature

In application, (1.6.1) is solved by reducing it to an equivalent system of first order ODEs and appropriate numerical techniques are used to solve the resulting system. These methods require more human effort and consequently enormous computer Central Processing Unit (CPU) time. This is even more involving if the programme code has subroutines Adesanya, Momoh, Alkali and Tahir (2012). For these and other reasons, these techniques are not efficient in general. Other methods of solutions to fourth order BVPs in literature include variational iteration method for fourth order initial Value Problems (IVPs), Adesanya *et al.*, (2012), Hussain, Ismail and Senua (2015) and (2016), Collocation method, Costabile and Napoli (2015), Sextic spline method for fourth order BVPs, Rashidinia and

Mohammadi (2010), just to mention but few.

For fifth order BVPs, several numerical methods including spectral Galerkin and collocation, decomposition, sixth order B-spline have been developed for solving fifth-order boundary value problems. Wazwaz (2001), developed a decomposition method for numerical solution of fifth-order boundary value problems. Shen (2007) solved fifth-order boundary value problems using the homotopy perturbation method. Ghazala and Hamood (2013) developed Searching the Least Value method for the solution of fifth order boundary value problems. Other authors who have worked on solution of fifth order BVPs include Gamel (2007) where he analyzed Sinc-Galerkin method for the solution of fifth-order boundary value problems with two-point boundary conditions; Taiwo, Adewumi and Ogunlaran (2015) developed an efficient numerical methods for solution of nonlinear higher-order boundary value problems. Other methods for handling fifth order BVPs, just to mention few, are in Juan (2009); Zhao (2009); Nazreen and AbdulMajid (2016).

As earlier stated, a wide range of numerical methods have been used to solve higher order PDEs with Dirichlet and Neumann boundary conditions that arise in engineering and sciences. For example, the Euler-Bernoulli equation is an approximate equation for a long and slender beam, whose solution is a transverse displacement of the beam from an initially horizontal position. This forms a fourth order parabolic PDEs, and have been solved with the use of Sextic spline method, Rashidinia and Mohammadi (2010). A typical fourth order parabolic PDE is described in Mittal and Jain, (2011), as follows:

$$y_{tt} + y_{xxxx} = G(x, t), \quad (2.2.1)$$

subject to appropriate boundary conditions.

The method of line was used in Saucez, *et al.*, (2014), to solve problems describing nonlinear wave phenomena, like the fourth order "good" Boussinesq equation. Other methods used include Quintic B-spline for the numerical solution of the "good" Boussinesq equation, B-splines methods with redefined basis functions for solving fourth order parabolic partial differential equations, Numerov type variable mesh approximations for

1D unsteady quasi-linear biharmonic problem, Mohanty and Kaur (2017), a Not-a-Knot meshless method using radial basis functions and predictor-corrector scheme, Shokri and Dehghan (2010), Quintic B-spline for the numerical solution of the "good" Boussinesq equation, Siddiqi and Arshed (2014), High accuracy implicit variable mesh methods for numerical study of special types of fourth order nonlinear parabolic equations, Mohanty and Kaur, (2016), and B-splines methods with redefined basis functions, Mittal and Jain (2011) and many others in literature.

The standard form of the "good" Boussinesq equation was described in Shokri and Dehghan (2010) as follows:

$$y_{tt} = y_{xx} + qy_{xxxx} + y_{xx}^2 \quad (2.2.2)$$

subject to appropriate boundary conditions, where $q = 1$ or -1 . In El-Zoheiry (2003), the "good" Boussinesq equation was studied numerically using an iterative implicit finite-difference scheme.

Another class of important fourth order PDEs were considered in Mohanty and Kaur (2017) and described as fourth-order quasi-linear PDEs are of the form

$$Q(x, t, y, y_{xx})y_{xxxx} + y_t = G(x, t, y, y_x, y_{xx}, y_{xxx}), \quad (2.2.3)$$

subject to appropriate boundary conditions, where $(x, t) \in \Theta \equiv \{(x, t) | a < x < b, t > 0\}$, a, b are real constants.

A different class of fourth order PDE is the Kuramoto-Sivashinsky equation (Sirajul *et al.*, 2010), of the form

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxx} + \eta u_{xxxx} = 0, \quad a \leq x \leq b, \quad t \geq 0 \quad (2.2.4)$$

where μ, ν and η are real constants. The numerical solution of the generalized Kuramoto-Sivashinsky equation (2.2.4) was presented in Sirajul, Nagina, Tirmizi and Usman (2010) using meshless Method of Lines (MOL). In this method, the spatial derivatives were ap-

proximated by Radial Basis Functions (RBFs) where no mesh is required for discretization of the problem domain. Only a set of scattered nodes is required to approximate the solution. Similarly, in Marjan, Sirajul and Siraj-ul (2009), RBFs based mesh-free method was implemented to find numerical solution of (2.2.4).

The fifth order KdV equation of the types

$$u_t - u_{xxxxx} = F(x, t, u, u^2, u_x, u_{xx}, u_{xxx}), \quad a \leq x \leq b, \quad t \geq 0 \quad (2.2.5)$$

was solved by the He's Variational Iteration Method Miansari, Barari and Ganji (2009), Homotopy analysis method Abbasbandy and Zakaria (2008), Group analysis method was applied in Wang, Kara, Fakhar, Guzman and Biswas (2016), Dual-Petrov-Galerkin method in Yuan, Shen and Wu (2008). Multiple-soliton solutions for the fifth-order Caudrey-Dodd-Gibbonv (CDG) equation Wazwaz (2008); Modebei *et al.*, (2015) of the type

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xxx} + 180u^2u_x = 0, \quad x \in [-1, 1], \quad t \in [0, T] \quad (2.2.6)$$

were obtained. The more general form of (2.2.6) is given as

$$u_t + au^2u_x + bu_xu_{xx} + cuu_{xxx} + du_{xxxxx} = 0 \quad (2.2.7)$$

where a , b , c and d are constants. For $a = 30$, $b = 30$, $c = 10$ and $d = 1$, equation (2.2.7) is called the Lax's fifth-order KdV equation and for $a = 45$, $b = 15$, $c = 15$ and $d = 1$, (2.2.7) is the Sawada-Kotera equation, Wazwaz, (2006). Mohyud-Din, Negahdary and Usman (2012) used meshless method for solution (2.2.7) by method of lines. This method uses Multiquadric (MQ), Inverse Multiquadric (IMQ) and Gaussian (GA) as Radial basis function (RBF) for spatial derivatives and Runge-Kutta method as a time integrator. Homotopy Perturbation Transform Method (HPTM) was also used by Goswami *et al.* (2017) for solving certain class of (2.2.7).

Another fifth order PDE of the form

$$u_t + f(u)_x + \sigma u_{xxxxx} = 0 \quad (2.2.8)$$

called a time-dependent biharmonic equation was also solved by Cheng and Chi-Wang (2008) using the Discontinuous Galerkin method. All these equations are considered in the numerical section of this thesis.

CHAPTER THREE: METHODOLOGY

3.1 Introduction

This chapter describes the techniques employed in the development of continuous implicit 5-step methods for the solutions of BVPs of fourth and fifth order ODE as well as fourth and fifth order PDEs. This basically involves approximating the exact solution $y(x)$ of (2.1.1) that is, in the partition $\pi_N = [a = x_0 < x_1 < \dots < x_N = b]$ of the integration interval $[a, b]$ by trial solution of the form;

$$y(x) \approx p(x) = \sum_{i=0}^k \rho_i T^*(x) \quad (3.1.1)$$

where $\rho \in \mathbb{R}$, $y \in C^\infty(a, b)$ and $T^*(x)$ is the shifted Chebyshev polynomial of the first kind. The fourth and fifth derivatives are;

$$y^{(\mu)}(x) \approx p^{(\mu)}(x) = \sum_{i=4}^k \rho_i T^{*(\mu)}(x), \quad (\mu = 4, 5) \quad (3.1.2)$$

The trial solution (3.1.1) is interpolated at some selected points while (3.1.2) is collocated at some selected points according to Jator (2008), thereby resulting into a system of equations. This system is solved using a Computer Algebraic System (CAS) like Mathematica and the values of the coefficients ρ_i , $i = 0, 1, \dots, k$. Then, the approximating polynomial (3.1.1) adopts the required continuous implicit 5-step methods respectively for $\mu = 4$ and 5. These adopted continuous implicit methods are finally evaluated at some points to obtain block methods which are called the block unification and boundary value methods respectively. Finally, these methods are implemented in its block form by written codes in *Mathematica 10.0* to solve problems in literature.

In the sections that follow, the derivation of four different continuous implicit k-step methods are outlined.

3.2 Derivation of the Methods

The exact solution $y(x)$ of (2.1.1) is approximated by seeking the continuous solution of the form

$$p(x) = \sum_{i=0}^{r+s-1} \rho_i T_i^*(x) \cong y(x) \quad (3.2.1)$$

with the μ th derivative given as

$$p^{(\mu)}(x) = \sum_{i=\mu}^{r+s-1} \rho_i T_i^{*(\mu)}(x) \cong y^{(\mu)}(x) \quad (3.2.2)$$

where $x \in [a, b]$, ρ_i 's are constants to be determined. $T_i^*(x)$, $i = 0(1)(r + s + 1)$, are shifted Chebyshev polynomials. The parameters r and s are respectively the number of interpolation points that satisfies $4 \leq r \leq \mu$ and the number of collocation points satisfying $0 < s \leq \mu + 1$, μ is the order of the differential equation.

Interpolating (3.2.1) at the points x_{n+i} ; $i = 0, 1, 2, \dots, r - 1$ and collocating (3.2.2) at the points x_{n+s} ; $s = 0, 1, 2, \dots, s - 1$, the following system of equations are obtained:

$$\begin{aligned} \sum_{i=0}^{r+s-1} \rho_i T_i^*(x) &= y_{n+i} & \square \\ \sum_{i=\mu}^{r+s-1} \rho_i T_i^{*(\mu)}(x) &= f_{n+i}, \quad x \in [0, k] & \square \end{aligned} \quad (3.2.3)$$

where the approximate values are given by $y_{n+i} \approx y(x_{n+i})$ and $y_{n+i}^{(\mu)} \approx f_{n+i} = f(x_{n+i}, y_{n+i}, \dots, y_{n+i}^{(\mu-1)})$.

The interpolation and collocation are defined in a single matrix as follows

$$\begin{array}{cccccccc}
 & T_0^*(x_n) & T_1^*(x_n) & \cdots & T_{r+s-1}^*(x_n) & & \rho_0 & y_n \\
 T_0^*(x_{n+1}) & T_1^*(x_{n+1}) & \cdots & T_{r+s-1}^*(x_{n+1}) & & \rho_1 & & y_{n+1} \\
 \cdot & \cdot & \cdot & \cdot & & \rho_2 & & \cdot \\
 T_0^*(x_{n+r-1}) & T_1^*(x_{n+r-1}) & \cdots & T_{r+s-1}^*(x_{n+r-1}) & & \rho_3 & = & y_{n+r-1} \\
 T_0^{*(\mu)}(x_n) & T_1^{*(\mu)}(x_n) & \cdots & T_{r+s-1}^{*(\mu)}(x_n) & & \rho_4 & & f_n \\
 T_0^{*(\mu)}(x_{n+1}) & T_1^{*(\mu)}(x_{n+1}) & \cdots & T_{r+s-1}^{*(\mu)}(x_{n+1}) & & \rho_5 & & f_{n+1} \\
 \cdot & \cdot & \cdots & \cdot & & \cdot & & \cdot \\
 T_0^{*(\mu)}(x_{n+s-1}) & T_1^{*(\mu)}(x_{n+s-1}) & \cdots & T_{r+s-1}^{*(\mu)}(x_{n+s-1}) & & \rho_{r+s-1} & & f_{n+s-1}
 \end{array} \quad (3.2.4)$$

for $T_n^*(x_{n+j})$ where $x_{n+j} \in [x_n, x_{n+k}]$.

3.2.1 Development of Methods for $\mu = 4$

The methods for $\mu = 4$, with the specifications $k = 5$, $r = 4$, $s = 6$, $T_n^*(x_{n+j})$ for $x_{n+j} \in [x_n, x_{n+5}]$ yield the system

$$\begin{array}{cccccccc}
 & T_0^*(x_n) & T_1^*(x_n) & \cdots & T_9^*(x_n) & & \rho_1 & y_n \\
 & T_0^*(x_{n+1}) & T_1^*(x_{n+1}) & \cdots & T_9^*(x_{n+1}) & & \rho_2 & y_{n+1} \\
 & T_0^*(x_{n+2}) & T_1^*(x_{n+2}) & \cdots & T_9^*(x_{n+2}) & & \rho_3 & y_{n+2} \\
 & T_0^*(x_{n+3}) & T_1^*(x_{n+3}) & \cdots & T_9^*(x_{n+3}) & & \rho_4 & y_{n+3} \\
 T_0^{*(iv)}(x_n) & T_1^{*(iv)}(x_n) & \cdots & T_9^{*(iv)}(x_n) & & \rho_5 & = & f_n \\
 T_0^{*(iv)}(x_{n+1}) & T_1^{*(iv)}(x_{n+1}) & \cdots & T_9^{*(iv)}(x_{n+1}) & & \rho_6 & = & f_{n+1} \\
 T_0^{*(iv)}(x_{n+2}) & T_1^{*(iv)}(x_{n+2}) & \cdots & T_9^{*(iv)}(x_{n+2}) & & \rho_7 & = & f_{n+2} \\
 T_0^{*(iv)}(x_{n+3}) & T_1^{*(iv)}(x_{n+3}) & \cdots & T_9^{*(iv)}(x_{n+3}) & & \rho_8 & = & f_{n+3} \\
 T_0^{*(iv)}(x_{n+4}) & T_1^{*(iv)}(x_{n+4}) & \cdots & T_9^{*(iv)}(x_{n+4}) & & \rho_9 & = & f_{n+4} \\
 T_0^{*(iv)}(x_{n+5}) & T_1^{*(iv)}(x_{n+5}) & \cdots & T_9^{*(iv)}(x_{n+5}) & & & & f_{n+5}
 \end{array} \quad (3.2.5)$$

where

$$\begin{aligned}
T_0^*(x) &= 1, T_1^*(x) = -1 + \frac{2x}{5}, T_2^*(x) = 1 - \frac{8x}{5} + \frac{8x^2}{25}, \\
T_3^*(x) &= -1 + \frac{18x}{5} - \frac{48x^2}{25} + \frac{32x^3}{125}, \\
T_4^*(x) &= 1 - \frac{32x}{5} + \frac{32x^2}{5} - \frac{256x^3}{125} + \frac{128x^4}{625}, \\
T_5^*(x) &= -1 + 10x - 16x^2 + \frac{224x^3}{25} - \frac{256x^4}{125} + \frac{512x^5}{3125}, \\
T_6^*(x) &= 1 - \frac{72x}{5} + \frac{168x^2}{5} - \frac{3584x^3}{125} + \frac{6912x^4}{625} - \frac{6144x^5}{3125} + \frac{2048x^6}{15625}, \\
T_7^*(x) &= -1 + \frac{98x}{5} - \frac{1568x^2}{25} + \frac{9408x^3}{125} - \frac{5376x^4}{125} + \frac{39424x^5}{3125} - \frac{28672x^6}{15625} + \frac{8192x^7}{78125}, \\
T_8^*(x) &= 1 - \frac{128x}{5} + \frac{2688x^2}{25} - \frac{21504x^3}{125} + \frac{16896x^4}{125} - \frac{180224x^5}{3125} + \frac{212992x^6}{15625} - \frac{131072x^7}{78125} + \frac{32768x^8}{390625}, \\
T_9^*(x) &= -1 + \frac{162x}{5} - \frac{864x^2}{5} + \frac{44352x^3}{125} - \frac{228096x^4}{625} + \frac{658944x^5}{3125} - \frac{1118208x^6}{15625} + \frac{221184x^7}{15625} - \frac{589824x^8}{390625} + \\
&\quad \frac{131072x^9}{1953125}, \\
T_0^{*(iv)}(x) &= T_1^{*(iv)}(x) = T_2^{*(iv)}(x) = T_3^{*(iv)}(x) = 0, \\
T_4^{*(iv)}(x) &= \frac{3072}{625}, T_5^{*(iv)}(x) = -\frac{6144}{125} + \frac{12288x}{625}, \\
T_6^{*(iv)}(x) &= \frac{165888}{625} - \frac{147456x}{625} + \frac{147456x^2}{3125}, \\
T_7^{*(iv)}(x) &= -\frac{129024}{125} + \frac{946176x}{625} - \frac{2064384x^2}{3125} + \frac{1376256x^3}{15625}, \\
T_8^{*(iv)}(x) &= \frac{405504}{125} - \frac{4325376x}{625} + \frac{15335424x^2}{3125} - \frac{22020096x^3}{15625} + \frac{11010048x^4}{78125}, \\
T_9^{*(iv)}(x) &= -\frac{5474304}{625} + \frac{15814656x}{625} - \frac{80510976x^2}{3125} + \frac{37158912x^3}{3125} - \frac{198180864x^4}{78125} + \frac{396361728x^5}{1953125}
\end{aligned}$$

Solving the above system, the values of the coefficients ρ_i , $i = 0, 1, \dots, 9$ are then substituted into (3.2.1) and after some simplification, the approximate polynomial (3.2.1) adopts the continuous form

$$p(x) = \sum_{i=0}^3 a_i(x)y_{n+i} + h^4 \sum_{i=0}^5 \beta_i(x)f_{n+i} \quad (3.2.6)$$

where the a_i 's, and β_i 's are continuous coefficients expressed as functions of t and given as:

$$a_0 = \frac{1}{48}(3 + 5t - 75t^2 - 125t^3), \quad a_1 = \frac{5}{16}(-1 - t + 25t^2 + 25t^3) \quad \square$$

$$a_2 = -\frac{5}{16}(-3 + 7t + 35t^2 + 25t^3), \quad a_3 = \frac{48}{16}(3 + 23t + 45t^2 + 25t^3) \quad \square$$

$$\beta_0 = \frac{185794560}{185794560}(36729 + 28779t - 1049700t^2 - 695500t^3 + 3543750t^4 - 708750t^5 - 6562500t^6 + 2812500t^7 + 3515625t^8 - 1953125t^9) \quad \square$$

$$\beta_1 = \frac{1}{37158912}(-430479 - 676299t + 10981980t^2 + 16840300t^3 - 5906250t^4 + 1968750t^5 + 10237500t^6 - 7312500t^7 - 2109375t^8 + 1953125t^9) \quad \square$$

$$\beta_2 = \frac{1}{18579456}(-533547 - 1478277t + 12637020t^2 + 37650500t^3 + 17718750t^4 - 17718750t^5 - 4462500t^6 + 9562500t^7 + 703125t^8 - 1953125t^9) \quad \square$$

$$\beta_3 = \frac{1}{18579456}(30933 - 129147t - 1474980t^2 + 2535100t^3 + 17718750t^4 + 17718750t^5 - 4462500t^6 - 9562500t^7 + 703125t^8 + 1953125t^9) \quad \square$$

$$\beta_4 = \frac{1}{37158912}(-27279 + 15051t + 901980t^2 - 309100t^3 - 5906250t^4 - 1968750t^5 + 10237500t^6 + 7312500t^7 - 2109375t^8 - 1953125t^9) \quad \square$$

$$\beta_5 = \frac{1}{185794560}(20601 - 1899t - 646500t^2 + 23500t^3 + 3543750t^4 + 708750t^5 - 6562500t^6 - 2812500t^7 + 3515625t^8 + 1953125t^9) \quad \square$$

(3.2.7)

where $t = \frac{1}{h} \cdot \frac{2x}{5} - 1$

In what follows, the Block Unification Method (BUM) is derived.

3.2.2 Development of the Block Unification Method (BUM)

Evaluating $p(x)$ in (3.2.6) at the points $x = x_{n+4}, x_{n+5}$, which implies $t = \frac{3}{5}, 1$, the following 5-step discrete LMMs are obtained

$$y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + h^4 \left(\frac{-1}{720}f_n + \frac{31}{180}f_{n+1} + \frac{79}{120}f_{n+2} \right) \quad \square$$

$$y_{n+5} = \frac{180}{180}y_{n+3} - \frac{720}{720}y_{n+2} + \frac{31}{720}f_{n+3} - \frac{1}{720}f_{n+4} - 4y_n + 15y_{n+1} - 20y_{n+2} + 10y_{n+3} + h^4 \left(\frac{1}{180}f_n + \frac{11}{16}f_{n+1} + \frac{101}{36}f_{n+2} \right) \quad \square$$

(3.2.8)

where $n = 0, 5, \dots, N - 5$.

From the transformation $x = \frac{(5ht+5)}{2}, \frac{2}{5} dx = hdt$, then the first derivatives of (3.2.7) yields

$$a_0^I = \frac{1}{h} \left[1 - \frac{5t}{8} - \frac{25t^2}{4} \right], \quad a_1^I = \frac{1}{h} \left[1 - \frac{1}{24} + \frac{25t}{4} + \frac{75t^2}{8} \right] \quad \square$$

$$a_2^I = \frac{1}{h} \left[-\frac{24}{8} - \frac{4}{8} - \frac{8}{8} \right], \quad a_3^I = \frac{1}{h} \left[\frac{23}{4096} + \frac{15t}{16384} + \frac{25t^2}{36864} - \frac{73728}{73728} \right] \quad \square$$

$$\beta_0^I = \frac{1}{h} \left[\frac{9593}{154828800} - \frac{3499t}{78173144} + \frac{6955t^2}{258048} + \frac{125t^3}{2064384} - \frac{125t^4}{2064384} + \frac{3125t^5}{2064384} + \frac{3125t^6}{2064384} \right] \quad \square$$

$$\beta_1^I = \frac{1}{h} \left[\frac{86016}{1720320} - \frac{2064384}{387072} + \frac{774144}{2048} - \frac{8192}{18432} + \frac{36864}{40625} - \frac{225433}{6779t} + 842015t^2 - 3125t^3 + 15625t^4 + 8125t^5 - 40625t^6 \right] \quad \square$$

$$\beta_2^I = \frac{1}{h} \left[\frac{129024}{30965760} - \frac{1032192}{28672} - \frac{1548288}{12288} - \frac{147456}{12288} + \frac{12288}{147456} + \frac{73728}{12288} - \frac{15625t^2}{86016} + \frac{390625t^3}{2064384} + 1882525t^2 + 3125t^3 - 15625t^4 - 10625t^5 + 53125t^6 \right] \quad \square$$

□

258048

2064384

(3.2.9)

15625^e

390625^e

— 129024 — 1032192

□

$$\beta_3^I = \begin{pmatrix} 1 & - & 43049 & \underline{24583t} & 126755t^2 & 3125t^3 & 15625t^4 & 10625t^5 & 53125t^6 \end{pmatrix}$$

$$\beta_4 = \frac{-15645 \pm \sqrt{15645^2 - 4 \cdot 390625}}{2 \cdot 390625} \Sigma \quad \square$$

$$\beta_5^t = 1 - \frac{211}{2155} - \frac{\sum}{t} + 235t^2 + 125t^3 + 125t^4 - 3125t^5 - 3125t^6$$

☐
☐

$$+ 15625 \ell' + 78125 \ell^{\Sigma}$$

□

$\begin{smallmatrix} 3 & 113 \\ -1, -5, -5, 5, 5, 1 \end{smallmatrix}$, the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
hy^n &= -\frac{11y_n}{6} + 3y_{n+1} - \frac{3y_{n+2}}{2} + \frac{y_{n+3}}{6} - h^4 \left[-\frac{937}{100800}f_n - \frac{19}{10080}f_{n+1} - \frac{599}{10080}f_{n+2} \right. \\
hy_{n+1}^j &= -\frac{728}{6}y_n + \frac{2240}{3}y_{n+1} - \frac{1360}{2}y_{n+2} + \frac{360}{1}y_{n+3} + h^4 \left[-\frac{100800}{1}f_n + \frac{105}{2809}f_{n+1} + \frac{10080}{43}f_{n+2} \right. \\
&\quad \left. - \frac{229}{30240}f_{n+3} + \frac{1}{33600}f_{n+5} - \frac{60480}{1008}f_{n+6} \right] \\
hy_{n+2}^j &= -\frac{1}{6}y_n + \frac{1}{2}y_{n+1} - \frac{1}{3}y_{n+2} + \frac{1}{6}y_{n+3} + h^4 \left[-\frac{169}{50400}f_n - \frac{311}{10080}f_{n+1} - \frac{353}{60480}f_{n+2} \right. \\
hy_{n+3}^j &= -\frac{61}{10080}f_{n+3} + \frac{17}{10080}f_{n+4} - \frac{11}{33600}f_{n+5} \\
hy_{n+4}^j &= -\frac{16}{15120}y_n + \frac{7}{60480}y_{n+1} - \frac{10}{3600}y_{n+2} + \frac{13}{3600}y_{n+3} - h^4 \left[-\frac{367400}{1}f_n + \frac{160}{2880}f_{n+1} + \frac{10080}{6847}f_{n+2} \right. \\
hy_{n+5}^j &= -\frac{13}{15120}y_n + \frac{7}{60480}y_{n+1} - \frac{10}{3600}y_{n+2} + \frac{13}{3600}y_{n+3} + h^4 \left[-\frac{31}{1}f_n + \frac{1663}{1}f_{n+1} + \frac{6847}{1}f_{n+2} \right. \\
&\quad \left. + \frac{60863}{1}f_{n+3} + \frac{2473}{1}f_{n+4} + \frac{2041}{1}f_{n+5} \right]
\end{aligned}$$

Differentiating (3.2.9) gives (3.2.10)

$$\begin{aligned}
a_0^j &= \frac{1}{h^2} \left[-\frac{1}{2} - \frac{5t}{2} \right] \\
a_1^j &= \frac{1}{h^2} \left[\frac{5}{2} + \frac{15t}{2} \right] \\
a_2^j &= \frac{1}{h^2} \left[-\frac{7}{2} - \frac{15t}{2} \right] \\
a_3^j &= \frac{1}{h^2} \left[\frac{3}{2} + \frac{5t}{2} \right] \\
\beta_0^j &= \frac{1}{h^2} \left[-\frac{1935360}{3499} - \frac{387072}{1391t} + \frac{2048}{75t^2} - \frac{2048}{25t^3} - \frac{18432}{3125t^4} + \frac{6144}{625t^5} + \frac{18432}{3125t^6} - \frac{129024}{15625t^7} \right] \\
\beta_1^j &= \frac{1}{h^2} \left[-\frac{6779}{71680} + \frac{168403t}{71680} - \frac{625t^2}{71680} + \frac{3125t^3}{71680} + \frac{8125t^4}{71680} - \frac{8125t^5}{71680} - \frac{3125t^6}{71680} + \frac{78125t^7}{71680} \right] \\
\beta_2^j &= \frac{1}{h^2} \left[-\frac{319617}{967680} + \frac{387072t}{967680} + \frac{2048t^2}{1875t^2} - \frac{18432t^3}{3125t^3} - \frac{6144t^4}{10625t^4} + \frac{6144t^5}{10625t^5} + \frac{6144t^6}{3125t^6} - \frac{129024t^7}{78125t^7} \right] \\
\beta_3^j &= \frac{1}{h^2} \left[-\frac{967680}{24583} + \frac{193536t}{25351t} + \frac{1024t^2}{1875t^2} + \frac{1024t^3}{3125t^3} - \frac{9216t^4}{10625t^4} - \frac{3072t^5}{10625t^5} + \frac{9216t^6}{3125t^6} + \frac{64512t^7}{78125t^7} \right] \\
\beta_4^j &= \frac{1}{h^2} \left[-\frac{501120}{387072} - \frac{387072}{387072} + \frac{2048}{75t^2} - \frac{18432}{3125t^3} + \frac{8125t^4}{3125t^4} + \frac{8125t^5}{625t^5} - \frac{6144}{3125t^6} - \frac{129024}{15625t^7} \right]
\end{aligned}$$

Evaluating $p^j(x)$ in (3.2.6) at the point $x = x_{n+i}$, $i = 0(1)5$, which implies $t = 1$, -1 , -5 , -5 , 5 , 5 , 1 , the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^2 y_n^{\text{II}} &= 2y_n - 5y_{n+1} + 4y_{n+2} - y_{n+3} + h^2 \left[\frac{1411}{2520} f_n + \frac{3091}{60480} f_{n+1} + \frac{2831}{6048} f_{n+2} \right. \\
&\quad \left. + \frac{143}{20160} f_{n+3} - \frac{1391}{4320} f_{n+4} + \frac{23}{1601} f_{n+5} \right] f_n - \frac{1}{1601} f_{n+1} + \frac{1}{1} f_{n+2} \\
h^2 y_{n+1}^{\text{II}} &= y_n - 2y_{n+1} + y_{n+2} + h^4 \left[-\frac{1}{73} f_n - \frac{1}{1601} f_{n+1} + \frac{1}{1} f_{n+2} \right. \\
&\quad \left. + \frac{143}{20160} f_{n+3} - \frac{1391}{4320} f_{n+4} + \frac{23}{1601} f_{n+5} \right] f_n - \frac{1}{1601} f_{n+1} + \frac{1}{1} f_{n+2} \\
h^2 y_{n+2}^{\text{II}} &= \frac{11}{y_{n+1}} f_{n+3} + \frac{11}{2y_{n+2}} f_{n+4} - \frac{11}{y_{n+3}} f_{n+5} + h^4 \left[\frac{11}{60480} f_n - \frac{53}{15120} f_{n+1} - \frac{773}{10080} f_{n+2} \right. \\
&\quad \left. - \frac{15320}{60480} f_{n+3} + \frac{60480}{60480} f_{n+4} \right] f_n - \frac{1}{15120} f_{n+1} - \frac{773}{10080} f_{n+2} \\
h^2 y_{n+3}^{\text{II}} &= -y_n + 4y_{n+1} - 5y_{n+2} + 2y_{n+3} + h^4 \left[-\frac{1}{720} f_n + \frac{10427}{60480} f_{n+1} + \frac{9901}{15120} f_{n+2} \right. \\
&\quad \left. + \frac{1120}{60480} f_{n+3} + \frac{7560}{60480} f_{n+4} \right] f_n + \frac{10427}{60480} f_{n+1} + \frac{9901}{15120} f_{n+2} \\
h^2 y_{n+4}^{\text{II}} &= \frac{10}{3780} y_{n+3} + \frac{7}{1344} y_{n+4} - \frac{8}{6048} y_{n+5} + h^4 \left[-\frac{179}{60480} f_n + \frac{3469}{10080} f_{n+1} + \frac{6421}{4320} f_{n+2} \right. \\
&\quad \left. + \frac{3791}{967680} f_{n+3} + \frac{121}{512} f_{n+4} - \frac{23}{1024} f_{n+5} \right] f_n + \frac{3469}{10080} f_{n+1} + \frac{6421}{4320} f_{n+2} \\
h^2 y_{n+5}^{\text{II}} &= -3y_n + 10y_{n+1} - 11y_{n+2} + 4y_{n+3} + h^4 \left[-\frac{11}{30240} f_n + \frac{29689}{60480} f_{n+1} + \frac{499}{213} f_{n+2} \right. \\
&\quad \left. + \frac{3791}{967680} f_{n+3} + \frac{121}{512} f_{n+4} - \frac{23}{1024} f_{n+5} \right] f_n + \frac{29689}{60480} f_{n+1} + \frac{499}{213} f_{n+2} \\
&\quad + \frac{3791}{967680} f_{n+3} + \frac{121}{512} f_{n+4} - \frac{23}{1024} f_{n+5} \\
\end{aligned}$$

Differentiating (3.2.11) gives

$$\begin{aligned}
a_0^{\text{II}} &= \frac{-1}{h^3}; \quad a_1^{\text{II}} = \frac{3}{h^3}; \quad a_2^{\text{II}} = \frac{-3}{h^3}; \quad a_3^{\text{II}} = \frac{1}{h^3} \\
\beta_0^{\text{II}} &= \frac{1}{h^3} \left[-\frac{1391}{967680} + \frac{15t}{512} - \frac{15t^2}{1024} - \frac{625t^3}{2304} + \frac{625t^4}{3072} + \frac{625t^5}{1536} - \frac{3125t^6}{9216} \right] \\
\beta_1^{\text{II}} &= \frac{1}{168403} - \frac{125t}{125t} + \frac{625t^2}{625t^2} + \frac{1625t^3}{1625t^3} - \frac{8125t^4}{8125t^4} - \frac{625t^5}{625t^5} + \frac{15625t^6}{15625t^6} \\
\beta_2^{\text{II}} &= \frac{1}{h^3} \left[\frac{75301}{987680} + \frac{375t}{256} - \frac{1875t^2}{512} - \frac{2125t^3}{1152} + \frac{10625t^4}{1536} + \frac{625t^5}{768} - \frac{15625t^6}{4608} \right] \\
\beta_3^{\text{II}} &= \frac{1}{h^3} \left[\frac{25351}{512} + \frac{375t}{3072} + \frac{1875t^2}{768} - \frac{2125t^3}{3072} - \frac{10625t^4}{512} + \frac{625t^5}{9216} + \frac{15625t^6}{9216} \right]
\end{aligned}$$

(3.2.13)

$$\begin{array}{ccccccc}
 483840 & & 256 & 5 & 11 & 1536 & 768 & 46 \\
 & & - & - & - & + & + & - & - & - \\
 5 & h^3 & 967680 & 512 & \overline{1024} & \overline{2304} & \overline{3072} & \overline{1536}
 \end{array}$$

$$\begin{array}{ccccccc}
 & & & 1 & 52 & & 08 \\
 & & & & & & \sum \\
 & & & & & & \square
 \end{array}$$

$$2$$

$$\beta^{\mathfrak{W}} = \underline{1}$$

$$\underline{3091} \quad \underline{125t} \quad 625t^2 \quad 1625t^3 \quad 8125t^4 \quad 625t^6 \quad 15625t^6$$

□

$$4 \quad h^3 \quad \cdot$$

Σ $\square$

$$\beta^{\mathbb{W}} = \frac{1}{125} - \frac{47}{125}t + \frac{15}{125}t^2 + 15t^3 - 625t^4 - 625t^5 + 625t^6 + \frac{3125}{125}t^6$$

□

$$\begin{array}{ccc} \underline{3} & & \underline{1} \underline{1} \underline{3} \end{array}$$

$-1, -5, -5, 5, 5, 1$, the following 5-step discrete LMMs are obtained:

which implies $t = -\frac{3}{5}, 1$, yields the following discrete 5-steps LMMs

$$\begin{aligned}
y_{n+4} &= 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + h^4 \cdot \frac{-1}{720}f_n + \frac{31}{180}f_{n+1} + \frac{79}{120}f_{n+2} \\
&\quad + \frac{31}{180}f_{n+3} - \frac{17}{720}f_{n+4} \quad \square \\
y_{n+5} &= \frac{97}{72}y_n + \frac{15}{6}y_{n+1} - \frac{20}{720}y_{n+2} + \frac{1}{5}y_{n+3} + h^4 \cdot \frac{-1}{720}f_n + \frac{11}{16}f_{n+1} + \frac{101}{36}f_{n+2} \\
&\quad + \frac{11}{16}f_{n+3} - \frac{101}{36}f_{n+4} \quad \square \\
hy_{n+4}^r &= -\frac{11}{12}y_n + \frac{7}{3}y_{n+1} - \frac{19}{509}y_{n+2} + \frac{13}{4}y_{n+3} + h^4 \cdot \frac{-671}{30240}f_n + \frac{169}{1660}f_{n+1} + \frac{12821}{60480}f_{n+2} \\
hy_{n+5}^r &= \frac{11}{12}y_n + \frac{7}{3}y_{n+1} - \frac{19}{509}y_{n+2} + \frac{13}{4}y_{n+3} + h^4 \cdot \frac{-671}{30240}f_n + \frac{169}{1660}f_{n+1} + \frac{12821}{60480}f_{n+2} \\
&\quad + \frac{11}{12}y_{n+4} - \frac{19}{509}y_{n+5} \quad \square \\
h^2 y_{n+4}^{rr} &= -2y_n + 7y_{n+1} - 8y_{n+2} + 3y_{n+3} + h^4 \cdot \frac{-60480}{179}f_n + \frac{10280}{3469}f_{n+1} + \frac{4180}{6421}f_{n+2} \\
&\quad + \frac{3780}{3791}f_{n+3} + \frac{121}{23}f_{n+4} - \frac{23}{60480}f_{n+5} \quad \square \\
h^2 y_{n+5}^{rr} &= -3y_n + 10y_{n+1} - 11y_{n+2} + 4y_{n+3} + h^4 \cdot \frac{-11}{30240}f_n + \frac{29689}{60480}f_{n+1} + \frac{499}{210}f_{n+2} \\
&\quad + \frac{58271}{30240}f_{n+3} + \frac{4561}{4320}f_{n+4} + \frac{271}{4032}f_{n+5} \quad \square \\
h^3 y_{n+4}^{rrr} &= -y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + h^4 \cdot \frac{-67}{12096}f_n + \frac{12049}{60480}f_{n+1} + \frac{22433}{30240}f_{n+2} \\
&\quad + \frac{35569}{30240}f_{n+3} + \frac{4873}{12096}f_{n+4} - \frac{883}{60480}f_{n+5} \quad \square \\
h^3 y_{n+5}^{rrr} &= -y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + h^4 \cdot \frac{-67}{12096}f_n + \frac{12049}{60480}f_{n+1} + \frac{22433}{30240}f_{n+2} \\
&\quad + \frac{35569}{30240}f_{n+3} + \frac{4873}{12096}f_{n+4} - \frac{883}{60480}f_{n+5} \quad \square \\
&\quad + \frac{18811}{60480}f_{n+3} + \frac{84299}{60480}f_{n+4} + \frac{19067}{60480}f_{n+5} \quad \square
\end{aligned} \tag{3.2.15}$$

where $n = 0, 5, \dots, N - 5$. The formulae in (3.2.15) constitute the main method. $\square$

For the additional method, by evaluating $p^l(x)$ in (3.2.6) at the points $x = x_i, i = 0(1)3$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}$, the following 5-step discrete LMM are obtained;

$$\begin{aligned}
hy_0^l &= -\frac{11y_0}{6} + 3y_1 - \frac{3y_2}{2} + \frac{y_3}{3} - h^4 \cdot \frac{-937}{100800}f_0 - \frac{19}{105}f_1 - \frac{599}{10080}f_2 \\
&\quad - \frac{1}{720}f_3 + \frac{2240}{33600}f_4 - \frac{1}{5}f_5 \quad \square \\
hy_1^l &= -y_0 - \frac{2}{1}y_1 + y_2 - \frac{1}{3}y_3 + h^4 \cdot \frac{5400}{33600}f_0 + \frac{2809}{60480}f_1 + \frac{43}{1008}f_2 \\
&\quad + \frac{229}{135}f_3 - \frac{1}{2880}f_4 - \frac{11}{151200}f_5 \quad \square \\
hy_2^l &= \frac{1}{6}y_0 - y_1 + \frac{1}{2}y_2 - \frac{1}{3}y_3 + h^4 \cdot \frac{-11}{302400}f_0 - \frac{31480}{60480}f_1 - \frac{353}{6048}f_2 \\
&\quad + \frac{229}{135}f_3 - \frac{1}{2880}f_4 - \frac{11}{151200}f_5 \quad \square \\
hy_3^l &= -\frac{1}{3}y_0 + \frac{7}{2}y_1 - \frac{53}{3}y_2 + \frac{11}{6}y_3 + h^4 \cdot \frac{-41}{50400}f_0 + \frac{173}{2880}f_1 + \frac{11}{60}f_2 \\
&\quad + \frac{61}{10080}f_3 + \frac{17}{10080}f_4 - \frac{11}{33600}f_5 \quad \square
\end{aligned} \tag{3.2.16}$$

Also by evaluating $p^u(x)$ in (3.2.6) at the points $x = x_i, i = 0(1)3$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}$, the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^2 y_0^{ll} &= 2y_0 - 5y_1 + 4y_2 - y_3 + h^2 \left[\frac{1411}{2520} f_0 + \frac{3091}{60480} f_1 + \frac{2831}{6048} f_2 \right. \\
&\quad \left. + \frac{143}{60480} f_3 - \frac{1391}{6048} f_4 + \frac{23}{6048} f_5 \right] - \frac{1601}{20160} f_0 - \frac{1}{4320} f_1 + \frac{1}{30240} f_2 \\
h^2 y_1^{ll} &= \frac{4320}{15120} f_3 + \frac{10080}{60480} f_4 + \frac{30240}{60480} f_5 - \frac{53}{15120} f_0 - \frac{773}{10080} f_1 - \frac{773}{10080} f_2 \\
h^2 y_2^{ll} &= \frac{53}{720} f_3 + \frac{11}{60480} f_4 + \frac{11}{60480} f_5 - \frac{1}{720} f_0 + \frac{10427}{60480} f_1 + \frac{9901}{15120} f_2 \\
&\quad + \frac{107}{1120} f_3 - \frac{37}{7560} f_4 + \frac{11}{60480} f_5 - \frac{1}{720} f_0 + \frac{10427}{60480} f_1 + \frac{9901}{15120} f_2
\end{aligned} \tag{3.2.17}$$

Similarly, by evaluating $p^{ll}(x)$ in (3.2.6) at the points $x = x_i$, $i = 0(1)3$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}$, the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^3 y_0^{rr} &= -y_0 + 3y_1 - 3y_2 + y_3 - h^3 \left[\frac{19151}{60480} f_0 - \frac{73967}{60480} f_1 + \frac{1261}{6048} f_2 \right. \\
&\quad \left. - \frac{30240}{7439} f_3 + \frac{60480}{5549} f_4 - \frac{60480}{883} f_5 \right] \\
h^3 y_1^{rr} &= -y_0 + 3y_1 - 3y_2 + y_3 + h^4 \left[\frac{799}{60480} f_0 - \frac{14033}{60480} f_1 - \frac{10453}{30240} f_2 \right. \\
&\quad \left. + \frac{30240}{2683} f_3 - \frac{60480}{1717} f_4 + \frac{60480}{251} f_5 \right] \\
h^3 y_2^{rr} &= -y_0 + 3y_1 - 3y_2 + y_3 + h^4 \left[\frac{16096}{60480} f_0 + \frac{60480}{12721} f_1 + \frac{30240}{11009} f_2 \right. \\
&\quad \left. - \frac{547}{60480} f_3 + \frac{1517}{60480} f_4 - \frac{211}{60480} f_5 \right] \\
h^3 y_3^{rr} &= -y_0 + 3y_1 - 3y_2 + y_3 + h^4 \left[\frac{16096}{60480} f_0 + \frac{60480}{12721} f_1 + \frac{30240}{11009} f_2 \right. \\
&\quad \left. - \frac{547}{60480} f_3 + \frac{1517}{60480} f_4 - \frac{211}{60480} f_5 \right]
\end{aligned} \tag{3.2.18}$$

The formula (3.2.15), (3.2.17), and (3.2.18) together form the additional methods.

Remark 3.2.2. The formula (3.2.15), (3.2.16), (3.2.17), and (3.2.18) together form the Boundary Value Method (BVM) which is used to solve (2.1.1) for $\mu = 4$ numerically with appropriate boundary conditions.

Remark 3.2.3. For both Block Unification Method and the Boundary Value Method, If IVP is to be solved, it would be applied sequentially on block of intervals of the form $[x_n, x_{n+5}]$, $n = 0, 5, \dots, N - 5$, where N , the number of subintervals, must be a multiple

of 5 in order to reach the final point $x_N = b$. In case of a BVP, all the formulae in each case of the BUM and BVM respectively are considered for $n = 0, 5, \dots, N - 5$. This

results in a system of $4N$ equations, which together with the four boundary conditions leads to a system of $4N+4$ equations in $4N+4$ unknowns $y_j, y_j^I, y_j^{II}, y_j^{III}$ for $j = 0(1)N$.

3.2.4 Development of Methods for $\mu = 5$

The methods for $\mu = 5$, with the following specifications $k = 5, r = 5, s = 6$, lead to the following system:

$$\begin{array}{ccccccc}
 T_0^*(x_n) & T_1^*(x_n) & \cdots & T_1^{*0}(x_n) & \rho_0 & y_n \\
 T_0^*(x_{n+1}) & T_1^*(x_{n+1}) & \cdots & T_1^{*0}(x_{n+1}) & \rho_1 & y_{n+1} \\
 T_0^*(x_{n+2}) & T_1^*(x_{n+2}) & \cdots & T_1^{*0}(x_{n+2}) & \rho_2 & y_{n+2} \\
 T_0^*(x_{n+3}) & T_1^*(x_{n+3}) & \cdots & T_1^{*0}(x_{n+3}) & \rho_3 & y_{n+3} \\
 T_0^*(x_{n+4}) & T_1^*(x_{n+4}) & \cdots & T_1^{*0}(x_{n+4}) & \rho_4 & y_{n+4} \\
 T_0^{*(v)}(x_n) & T_1^{*(v)}(x_n) & \cdots & T_{10}^{*(v)}(x_n) & \rho_5 & f_n \\
 T_0^{*(v)}(x_{n+1}) & T_1^{*(v)}(x_{n+1}) & \cdots & T_{10}^{*(v)}(x_{n+1}) & \rho_6 & f_{n+1} \\
 T_0^{*(v)}(x_{n+2}) & T_1^{*(v)}(x_{n+2}) & \cdots & T_{10}^{*(v)}(x_{n+2}) & \rho_7 & f_{n+2} \\
 T_0^{*(v)}(x_{n+3}) & T_1^{*(v)}(x_{n+3}) & \cdots & T_{10}^{*(v)}(x_{n+3}) & \rho_8 & f_{n+3} \\
 T_0^{*(v)}(x_{n+4}) & T_1^{*(v)}(x_{n+4}) & \cdots & T_{10}^{*(v)}(x_{n+4}) & \rho_9 & f_{n+4} \\
 T_0^{*(v)}(x_{n+5}) & T_1^{*(v)}(x_{n+5}) & \cdots & T_{10}^{*(v)}(x_{n+5}) & \rho_{10} & f_{n+5}
 \end{array} \quad (3.2.19)$$

where

$$\begin{aligned}
 T_0^*(x) &= 1, T_1^*(x) = -1 + \frac{2x}{5}, T_2^*(x) = 1 - \frac{8x}{5} + \frac{8x^2}{25}, \\
 T_3^*(x) &= -1 + \frac{18x}{5} - \frac{48x^2}{25} + \frac{32x^3}{125}, T_4^*(x) = 1 - \frac{32x}{5} + \frac{32x^2}{5} - \frac{256x^3}{125} + \frac{128x^4}{625}, \\
 T_5^*(x) &= -1 + 10x - 16x^2 + \frac{224x^3}{25} - \frac{256x^4}{125} + \frac{512x^5}{3125}, \\
 T_6^*(x) &= 1 - \frac{72x}{5} + \frac{168x^2}{5} - \frac{3584x^3}{125} + \frac{6912x^4}{625} - \frac{6144x^5}{3125} + \frac{2048x^6}{15625}, \\
 T_7^*(x) &= -1 + \frac{98x}{5} - \frac{1568x^2}{25} + \frac{9408x^3}{125} - \frac{5376x^4}{125} + \frac{39424x^5}{3125} - \frac{28672x^6}{15625} + \frac{8192x^7}{78125}, \\
 T_8^*(x) &= 1 - \frac{128x}{5} + \frac{2688x^2}{25} - \frac{21504x^3}{125} + \frac{16896x^4}{125} - \frac{180224x^5}{3125} + \frac{212992x^6}{15625} - \frac{131072x^7}{78125} + \frac{32768x^8}{390625}, \\
 T_9^*(x) &= -1 + \frac{162x}{5} - \frac{864x^2}{5} + \frac{44352x^3}{125} - \frac{228096x^4}{625} + \frac{658944x^5}{3125} - \frac{1118208x^6}{15625} + \frac{221184x^7}{15625} - \frac{589824x^8}{390625} + \frac{131072x^9}{1953125}, \\
 T_{10}^{*0}(x) &= 1 - 40x + 264x^2 - \frac{16896x^3}{25} + \frac{109824x^4}{125} - \frac{2050048x^5}{3125} + \frac{186368x^6}{625} - \frac{262144x^7}{3125} + \frac{1114112x^8}{78125} - \frac{524288x^9}{390625} + \frac{524288x^{10}}{9765625},
 \end{aligned}$$

$$\begin{aligned}
T_0^{*(v)}(x) &= T_2^{*(v)}(x) = T_3^{*(v)}(x) = T_4^{*(v)}(x) = 0, \\
T_5^{*(v)}(x) &= \frac{12288}{625}, \quad T_6^{*(v)}(x) = \frac{-147456}{625} + \frac{294912x}{3125}, \\
T_7^{*(v)}(x) &= \frac{946176}{625} - \frac{4128768x}{3125} + \frac{4128768x^2}{15625}, \\
T_8^{*(v)}(x) &= \frac{-4325376}{625} + \frac{30670848x}{3125} - \frac{66060288x^2}{15625} + \frac{44040192x^3}{78125}, \\
T_9^{*(v)}(x) &= \frac{15814656}{625} - \frac{161021952x}{3125} + \frac{111476736x^2}{3125} - \frac{792723456x^3}{78125} + \frac{396361728x^4}{390625}, \\
T_{10}^{*(v)}(x) &= \frac{-49201152}{625} + \frac{26836992x}{125} - \frac{132120576x^2}{625} + \frac{1497366528x^3}{15625} - \frac{1585446912x^4}{78125} + \frac{3170893824x^5}{1953125}
\end{aligned}$$

As illustrated earlier, solving the above system, the values of the coefficients ρ_i , $i = 0, 1, \dots, 10$ are obtained. The ρ_i 's are afterward substituted into (3.2.1) and after some simplification, the approximate polynomial (3.2.1) takes the continuous form

$$p(x) = \sum_{i=0}^4 a_i(x)y_{n+i} + h^5 \sum_{i=0}^5 \beta_i(x)f_{n+i} \quad (3.2.20)$$

where the a_i 's, and β_i 's are continuous coefficients expressed as function of t as follows

$$\begin{aligned}
a_0 &= \frac{1}{384}(9 - 250t^2 + 625t^4) & \square \\
a_1 &= -\frac{5}{96}(3 - 2t - 80t^2 + 50t^3 + 125t^4) & \square \\
a_2 &= \frac{5}{64}(9 - 36t - 70t^2 + 100t^3 + 125t^4) & \square \\
a_3 &= -\frac{5}{96}(-9 - 54t - 20t^2 + 150t^3 + 125t^4) & \square \\
a_4 &= \frac{5}{384}(-3 - 8t + 70t^2 + 200t^3 + 125t^4) & \square \\
\beta_0 &= -\frac{1}{148635648}(1 - 5t)^2(2079 + 9540t - 14154t^2 - 45040t^3 + 35200t^4 \\
&\quad + 60500t^5 - 38750t^6 - 25000t^7 + 15625t^8) & \square
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= \frac{1}{148635648}(166131 - 90774t - 4616055t^2 + 2718600t^3 + 11452750t^4 \\
&\quad - 11812500t^5 + 3281250t^6 + 14625000t^7 - 9140625t^8 - 2343750t^9 + 1953125t^{10}) \quad \square \\
\beta_2 &= \frac{1}{74317824}(626913 + 475218t - 17774745t^2 - 13287800t^3 + 53710250t^4 \\
&\quad + 35437500t^5 - 29531250t^6 - 6375000t^7 + 11953125t^8 + 781250t^9 - 1953125t^{10}) \quad \square \\
\beta_3 &= \frac{1}{74317824}(171423 + 475218t - 4401255t^2 - 13287800t^3 + 1729750t^4 \\
&\quad + 35437500t^5 + 29531250t^6 - 6375000t^7 - 11953125t^8 + 781250t^9 + 1953125t^{10}) \quad \square \\
\beta_4 &= \frac{1}{148635648}(-20979 - 90774t + 584055t^2 + 2718600t^3 - 1372750t^4 \\
&\quad - 11812500t^5 - 3281250t^6 + 14625000t^7 + 9140625t^8 - 2343750t^9 - 1953125t^{10}) \quad \square \\
\beta_5 &= \frac{1}{148635648}(1 + 5t)^2(2079 - 9540t - 14154t^2 + 45040t^3 + 35200t^4 \\
&\quad - 60500t^5 - 38750t^6 + 25000t^7 + 15625t^8) \quad \square
\end{aligned} \tag{3.2.21}$$

where $t = \frac{1}{h} \cdot \frac{2x}{5} - 1$.

In what follows, the BUM is derived.

3.2.5 Development of the Block Unification Method (BUM)

By evaluating $p(x)$ in (3.2.20) at the point $x = x_{n+5}$, which implies $t = 1$, the following 5-step discrete LMM is obtained

$$y_{n+5} = y_n - 5y_{n+1} + 10y_{n+2} - 10y_{n+3} + 5y_{n+4} + h^5 \left[\frac{1}{24}f_{n+1} + \frac{11}{24}f_{n+2} + \frac{11}{24}f_{n+3} + \frac{1}{24}f_{n+4} \right] \tag{3.2.22}$$

The formula (3.2.22) gives the main method.

From the transformation $x = \frac{(5t+5)}{2}$, $\frac{2}{5}dx = hdt$, the first derivatives of (3.2.21) yields

$$\begin{aligned}
\alpha_0^1 &= \frac{25}{48h}(-t + 5t^3), \quad \alpha_1^1 = \frac{1}{24h}(1 + 80t - 75t^2 - 250t^3), \quad \square \\
\alpha_2^1 &= \frac{8h}{24h}(-9 - 35t + 75t^2 + 125t^3), \quad \alpha_3^1 = \frac{24h}{24h}(27 + 20t - 225t^2 - 250t^3), \quad \square \\
\alpha_4^1 &= \frac{1}{48h}(-2 + 35t + 150t^2 + 125t^3), \quad \square \\
\beta_0^1 &= \frac{185794560h}{185794560h}(5625 + 57579t - 502500t^2 - 263500t^3 + 3543750t^4 \\
&\quad - 708750t^5 - 6562500t^6 + 2812500t^7 + 3515625t^8 - 1953125t^9), \quad \square \\
\beta_1^1 &= \frac{1}{185794560h}(-45387 - 4616055t + 4077900t^2 + 22905500t^3 - 29531250t^4 \\
&\quad + 9843750t^5 + 51187500t^6 - 36562500t^7 - 10546875t^8 + 9765625t^9), \quad \square \\
\beta_2^1 &= \frac{1}{92897280h}(237609 - 17774745t - 19931700t^2 + 107420500t^3 + 88593750t^4 \\
&\quad - 88593750t^5 - 22312500t^6 + 47812500t^7 + 3515625t^8 - 9765625t^9), \quad \square \\
\beta_3^1 &= \frac{1}{92897280h}(237609 - 4401255t - 19931700t^2 + 3459500t^3 + 88593750t^4 \\
&\quad + 88593750t^5 - 22312500t^6 - 47812500t^7 + 3515625t^8 + 9765625t^9), \quad \square \\
\beta_4^1 &= \frac{1}{185794560h}(-45387 + 584055t + 4077900t^2 - 2745500t^3 - 29531250t^4 \\
&\quad - 9843750t^5 + 51187500t^6 + 36562500t^7 - 10546875t^8 - 9765625t^9), \quad \square \\
\beta_5^1 &= \frac{1}{185794560h}(5625 - 57579t - 502500t^2 + 263500t^3 + 3543750t^4 \\
&\quad + 708750t^5 - 6562500t^6 - 2812500t^7 + 3515625t^8 + 1953125t^9) \quad \square
\end{aligned}
\tag{3.2.23}$$

By evaluating $p^i(x)$ in (3.2.20) at the point $x = x_{n+i}$, $i = 0(1)5$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1$ the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
hy^f &= -\frac{25y_n}{4} - 4y_{n+1} - 3y_{n+2} + \frac{4y_{n+3}}{4} - \frac{y_{n+4}}{4} + h^5 \left(-\frac{f_n}{4} + \frac{107f_{n+1}}{4} + \frac{199f_{n+2}}{4} \right) \quad \square \\
n &+ \frac{560}{37n+3} + \frac{10080}{19f_{n+4}} - \frac{3360}{f_{n+5}} \sum \quad 3 \quad 4 \\
n+1 &+ \frac{4}{4} - \frac{5y_{n+1}}{6} + \frac{3y_{n+2}}{2} - \frac{y_{n+3}}{2} + \frac{y_{n+4}}{12} + h^5 \left(-\frac{3360}{f_n} - \frac{1440}{f_{n+1}} - \frac{1680}{361f_{n+2}} \right) \quad \square \\
hy^f &= -\frac{f_{n+3}}{4} - \frac{f_{n+4}}{4} + \frac{f_{n+5}}{4} \quad \square \\
hy^{n+2} &= \frac{12}{480} - \frac{2y_{n+1}}{1680} + \frac{2y_{n+3}}{10080} - \frac{y_{n+4}}{12} + h^5 \left(-\frac{10080}{f_n} + \frac{12520}{11f_{n+1}} + \frac{24908}{25f_{n+2}} \right) \quad \square \\
hy^{r+3} &= \pm \frac{11f_{n+3}}{224} + \frac{10080}{1120} - \frac{3y_{n+2}}{1008} + \frac{5y_{n+3}}{4} + \frac{y_{n+4}}{4} - h^5 \left(-\frac{f_{n+1}}{4} - \frac{341f_{n+2}}{4} \right) \quad \square \\
n &+ \frac{13}{4} - \frac{2f_{n+4}}{3} + \frac{2f_{n+5}}{3} - \frac{4y_{n+3}}{4} + \frac{25y_{n+4}}{12} + h^5 \left(-\frac{280}{f_n} + \frac{10080}{11f_{n+1}} + \frac{9f_{n+2}}{80} \right) \quad \square \\
hy^f &= \frac{y_n}{4} - \frac{5y_{n+1}}{3} + \frac{3y_{n+2}}{2} - \frac{y_{n+3}}{2} + \frac{y_{n+4}}{12} + h^5 \left(-\frac{f_n}{4} + \frac{149f_{n+1}}{4} + \frac{9679f_{n+2}}{4} \right) \quad \square \\
n+4 &+ \frac{5040}{10080} - \frac{672}{180} - \frac{3360}{3360} \sum \quad 6 \quad 12 \quad 3360 \quad 1680 \quad 10080 \\
hy^f &= \frac{25y_n}{10080} - \frac{5y_{n+1}}{180} + \frac{3y_{n+2}}{3360} + \frac{77y_{n+4}}{4} + h^5 \left(-\frac{f_n}{4} + \frac{149f_{n+1}}{4} + \frac{9679f_{n+2}}{4} \right) \quad \square \\
n+5 &+ \frac{10819f_{n+3}}{4} + \frac{29f_{n+4}}{4} + \frac{f_{n+5}}{4} \quad \square \\
\end{aligned} \tag{3.2.24}$$

Differentiating (3.2.23) yields the following

$$\begin{aligned}
a_0^{\text{II}} &= \frac{5}{24h^2} (-1 + 15t^2), \quad a_1^{\text{II}} = \frac{1}{6h^2} (8 - 15t - 75t^2), \quad a_2^{\text{II}} = \frac{1}{4h^2} (-7 + 30t + 75t^2) \quad \square \\
a_3^{\text{II}} &= \frac{1}{6h^2} (2 - 45t - 75t^2), \quad a_4^{\text{II}} = \frac{1}{24h^2} (7 + 60t + 75t^2), \quad \square \\
\beta_0^{\text{II}} &= \frac{1}{15482880h^2} (19193 - 335000t - 263500t^2 + 4725000t^3 \\
&\quad - 1181250t^4 - 13125000t^5 + 6562500t^6 + 9375000t^7 - 5859375t^8), \quad \square \\
\beta_1^{\text{II}} &= \frac{1}{30965760h^2} (-307737 + 543720t + 4581100t^2 - 7875000t^3 \\
&\quad + 3281250t^4 + 20475000t^5 - 17062500t^6 - 5625000t^7 + 5859375t^8), \quad \square \\
\beta_2^{\text{II}} &= \frac{1}{15482880h^2} (-1184983 - 2657560t + 21484100t^2 + 23625000t^3 \\
&\quad - 29531250t^4 - 8925000t^5 + 22312500t^6 + 1875000t^7 - 5859375t^8), \quad \square \\
\beta_3^{\text{II}} &= \frac{1}{15482880h^2} (-293417 - 2657560t + 691900t^2 + 23625000t^3 \\
&\quad + 29531250t^4 - 8925000t^5 - 22312500t^6 + 1875000t^7 + 5859375t^8), \quad \square \\
\beta_4^{\text{II}} &= \frac{1}{30965760h^2} (38937 + 543720t - 549100t^2 - 7875000t^3 \\
&\quad - 3281250t^4 + 20475000t^5 + 17062500t^6 - 5625000t^7 - 5859375t^8), \quad \square \\
\beta_5^{\text{II}} &= \frac{1}{15482880h^2} (-19193 - 335000t + 263500t^2 + 4725000t^3 \\
&\quad + 1181250t^4 - 13125000t^5 - 6562500t^6 + 9375000t^7 + 5859375t^8) \quad \square \\
\end{aligned} \tag{3.2.25}$$

By evaluating $p^{\text{II}}(x)$ in (3.2.20) at the points $x = x_{n+i}$, $i = 0(1)5$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1$ the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^2 y_n^{\text{II}} &= \frac{35y_n}{12} - \frac{26y_{n+1}}{3} + \frac{19}{2}y_{n+2} - \frac{14y_{n+3}}{3} + \frac{11y_{n+4}}{12} + h^5 \left[\frac{-887f_n}{100800} - \frac{21811f_{n+1}}{60480} \right. \\
&= \frac{11y_n}{12} - \frac{13079f_{n+2}}{576} + \frac{41f_{n+3}}{576} - \frac{209f_{n+4}}{30240} + \frac{23f_{n+5}}{30240} + h^5 \left[\frac{-23h^3f_n}{43200} + \frac{73f_{n+1}}{2240} + \frac{779f_{n+2}}{15120} \right. \\
h^2 y_{n+1}^{\text{II}} &= \frac{12y_n}{12} + \frac{4y_{n+1}}{3} - \frac{5y_{n+2}}{2} + \frac{4y_{n+3}}{3} - \frac{12y_{n+4}}{12} + h^5 \left[\frac{-302400}{100800} \frac{79f_n}{8640} - \frac{60480}{337} \frac{f_{n+1}}{108} - \frac{10080}{17} \frac{f_{n+2}}{108} \right. \\
&\quad + \frac{864}{7} \frac{f_{n+3}}{108} - \frac{8640}{11} \frac{f_{n+4}}{108} + \frac{100800}{17} \frac{f_{n+5}}{108} \Sigma \\
h^2 y_{n+2}^{\text{II}} &= -\frac{12y_n}{12} + \frac{y_{n+1}}{3} + \frac{y_{n+2}}{2} - \frac{5y_{n+3}}{2} + \frac{11y_{n+4}}{12} + h^5 \left[\frac{-17f_n}{100800} - \frac{19f_{n+1}}{60480} - \frac{5f_{n+2}}{720} \right. \\
&\quad - \frac{630}{23} \frac{f_{n+3}}{108} + \frac{60480}{127} \frac{f_{n+4}}{108} - \frac{302400}{79} \frac{f_{n+5}}{108} \Sigma \\
h^2 y_{n+3}^{\text{II}} &= \frac{11y_n}{12} - \frac{14y_{n+1}}{3} + \frac{19y_{n+2}}{2} - \frac{26y_{n+3}}{3} + \frac{35y_{n+4}}{12} + h^5 \left[\frac{302400}{79} \frac{f_n}{8640} + \frac{192}{7} \frac{f_{n+1}}{108} + \frac{30240}{12757} \frac{f_{n+2}}{108} \right. \\
h^2 y_{n+4}^{\text{II}} &= \frac{35y_n}{12} - \frac{11147f_{n+3}}{576} + \frac{113f_{n+4}}{576} - \frac{23f_{n+5}}{30240} + \frac{71y_{n+4}}{12} + h^5 \left[\frac{-23f_n}{43200} + \frac{7559f_{n+1}}{60480} + \frac{983f_{n+2}}{720} \right. \\
h^2 y_{n+5}^{\text{II}} &= \frac{11y_n}{12} - \frac{14y_{n+1}}{3} + \frac{19y_{n+2}}{2} - \frac{26y_{n+3}}{3} + \frac{35y_{n+4}}{12} + h^5 \left[\frac{-23f_n}{43200} + \frac{7559f_{n+1}}{60480} + \frac{983f_{n+2}}{720} \right. \\
&\quad + \frac{1672f_{n+3}}{100800} + \frac{29161f_{n+4}}{100800} + \frac{887f_{n+5}}{100800} \Sigma \\
&\quad + \frac{1672f_{n+3}}{100800} + \frac{29161f_{n+4}}{100800} + \frac{887f_{n+5}}{100800} \Sigma
\end{aligned} \tag{3.2.26}$$

Differentiating (3.2.25) yields the following

$$\begin{aligned}
\alpha_0^{\text{III}} &= \frac{5t}{2h^3}, \quad \alpha_1^{\text{III}} = \frac{-1-10t}{h^3}, \quad \alpha_2^{\text{III}} = \frac{3+15t}{h^3}, \quad \alpha_3^{\text{III}} = \frac{-3-10t}{h^3}, \quad \alpha_4^{\text{III}} = \frac{2+5t}{2h^3}, \\
\beta_0^{\text{III}} &= \frac{1}{387072h^3} (-335 + 527t - 14175t^2 + 4725t^3 + 65625t^4 \\
&\quad - 39375t^5 - 65625t^6 + 46875t^7) \\
\beta_1^{\text{III}} &= \frac{1}{1935360h^3} (13593 + 229055t - 590625t^2 + 328125t^3 + 2559375t^4 \\
&\quad - 2559375t^5 - 984375t^6 + 1171875t^7) \\
\beta_2^{\text{III}} &= \frac{1}{967680h^3} (-66439 + 1074205t + 1771875t^2 - 2953125t^3 - 1115625t^4 \\
&\quad + 3346875t^5 + 328125t^6 - 1171875t^7)
\end{aligned}$$

$$\begin{aligned}
\beta_3^{\mathbb{W}} &= \frac{1}{967680h^3} (-66439 + 34595t + 1771875t^2 + 2953125t^3 - 1115625t^4 \\
&\quad - 3346875t^5 + 328125t^6 + 1171875t^7) \quad \square \\
\beta_4^{\mathbb{W}} &= \frac{1}{1935360h^3} (-13593 + 27455t + 590625t^2 + 328125t^3 - 2559375t^4 \\
&\quad - 2559375t^5 + 984375t^6 + 1171875t^7) \quad \square \\
\beta_5^{\mathbb{W}} &= \frac{1}{387072h^3} (-335 + 527t + 14175t^2 + 4725t^3 - 65625t^4 \\
&\quad - 39375t^5 + 65625t^6 + 46875t^7) \quad \square \\
&\quad \square \\
&\quad \square
\end{aligned} \tag{3.2.27}$$

By evaluating $p^{\mathbb{W}}(x)$ in (3.2.20) at the points $x = x_{n+i}$, $i = 0(1)5$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1$ the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^3 y^{\mathbb{W}} &= -\frac{5y_n}{2} + 9y_{n+1} - 12y_{n+2} + 7y_{n+3} - \frac{3y_{n+4}}{2} + h^5 \left[\frac{277f_n}{4032} + \frac{4081f_{n+1}}{4320} + \frac{19433f_{n+2}}{30240} \right. \\
&\quad \left. + \frac{137f_{n+3}}{60480} - \frac{1061f_{n+4}}{60480} + \frac{17f_{n+5}}{60480} \right] \quad \square \\
h^3 y_{n+1}^{\mathbb{W}} &= -\frac{3y_n}{2} + 5y_{n+1} - 6y_{n+2} + 3y_{n+3} - \frac{y_{n+4}}{2} + h^5 \left[\frac{-17f_n}{6048} + \frac{31f_{n+1}}{1344} + \frac{3251f_{n+2}}{15120} \right. \\
&\quad \left. + \frac{49f_{n+3}}{4320} + \frac{41f_{n+4}}{10080} - \frac{41f_{n+5}}{60480} \right] \quad \square \\
h^3 y^{\mathbb{W}} &= -\frac{y_n}{15120} + \frac{y_{n+1}}{60480} - y_{n+2} + \frac{y_{n+3}}{60480} + h^5 \left[\frac{41f_n}{60480} - \frac{419f_{n+1}}{60480} - \frac{395f_{n+2}}{60480} \right. \\
&\quad \left. + \frac{67f_{n+3}}{2016} - \frac{13f_{n+4}}{1890} + \frac{41f_{n+5}}{60480} \right] \quad \square \\
h^3 y_{n+3}^{\mathbb{W}} &= \frac{y_n}{2} - 3y_{n+1} + 6y_{n+2} - 5y_{n+3} + \frac{3y_{n+4}}{2} + h^5 \left[\frac{1301f_{n+1}}{60480} + \frac{1523f_{n+2}}{7560} \right. \\
&\quad \left. + \frac{67f_{n+3}}{2016} - \frac{13f_{n+4}}{1890} + \frac{41f_{n+5}}{60480} \right] \quad \square \\
h^3 y^{\mathbb{W}} &= \frac{3y_n}{7560} - 7y_{n+1} + 12y_{n+2} - 9y_{n+3} + \frac{5y_{n+4}}{60480} + h^5 \left[\frac{-41f_n}{60480} + \frac{671f_{n+1}}{60480} + \frac{3019f_{n+2}}{60480} \right. \\
&\quad \left. + \frac{54083f_{n+3}}{30240} + \frac{4531f_{n+4}}{4320} + \frac{277f_{n+5}}{4032} \right] \quad \square \\
h^3 y_{n+5}^{\mathbb{W}} &= \frac{5y_n}{2} - 11y_{n+1} + 18y_{n+2} - 13y_{n+3} + \frac{7y_{n+4}}{2} + h^5 \left[\frac{17f_n}{6048} + \frac{5239f_{n+1}}{60480} + \frac{6323f_{n+2}}{5040} \right. \\
&\quad \left. + \frac{54083f_{n+3}}{30240} + \frac{4531f_{n+4}}{4320} + \frac{277f_{n+5}}{4032} \right] \quad \square \\
&\quad \square
\end{aligned}$$

Finally, differentiating (3.2.27) yield

(3.2.28)

$$a_0^{(iv)} = h^4, \quad a_1^{(iv)} = \frac{-4}{h^4}, \quad a_2^{(iv)} = \frac{6}{h^4}, \quad a_3^{(iv)} = \frac{-4}{h^4}, \quad a_4^{(iv)} = h^4 \quad \square$$

$$\beta_0^{(iv)} = \frac{1}{967680h^4}(-527 - 28350t + 14175t^2 + 262500t^3 - 196875t^4 - 393750t^5 + 328125t^6) \quad \square$$

$$\beta_1^{(iv)} = \frac{1}{967680h^4}(45811 - 236250t + 196875t^2 + 2047500t^3 - 2559375t^4 - 1181250t^5 + 1640625t^6) \quad \square$$

$$\beta_2^{(iv)} = \frac{1}{483840h^4}(214841 + 708750t - 1771875t^2 - 892500t^3 + 3346875t^4 + 393750t^5 - 1640625t^6) \quad \square$$

$$\beta_3^{(iv)} = \frac{1}{483840h^4}(6919 + 708750t + 1771875t^2 - 892500t^3 - 3346875t^4 + 393750t^5 + 1640625t^6) \quad \square$$

$$\beta_4^{(iv)} = \frac{1}{967680h^4}(-5491 + 236250t + 196875t^2 - 2047500t^3 - 2559375t^4 + 1181250t^5 + 1640625t^6) \quad \square$$

$$\beta_5^{(iv)} = \frac{1}{967680h^4}(527 + 28350t + 14175t^2 - 262500t^3 - 196875t^4 + 393750t^5 + 328125t^6) \quad \square$$

(3.2.29)

By evaluating $p^{(iv)}(x)$ in (3.2.20) at the points $x = x_{n+i}$, $i = 0(1)5$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1$ the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^4 y_n^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5 \left(\frac{-19097f_n}{60480} - \frac{81629f_{n+1}}{60480} \right) \quad \square \\
&\quad \frac{30240}{30240} \quad \frac{30240}{30240} \quad \frac{60480}{60480} \quad \frac{60480}{60480} \\
h^4 y_{n+1}^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5 \left(\frac{-3799f_{n+2}}{60480} - \frac{8591f_{n+3}}{60480} + \frac{5399f_{n+4}}{60480} - \frac{853f_{n+5}}{60480} \right) \quad \square \\
&\quad - \frac{3799f_{n+2}}{60480} + \frac{8591f_{n+3}}{60480} - \frac{5399f_{n+4}}{60480} + \frac{853f_{n+5}}{60480} \\
h^4 y_{n+2}^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5 \left(\frac{-20557f_{n+2}}{60480} - \frac{15311f_{n+3}}{60480} + \frac{18679f_{n+4}}{60480} - \frac{281f_{n+5}}{60480} \right) \quad \square \\
&\quad + \frac{20557f_{n+2}}{60480} - \frac{15311f_{n+3}}{60480} + \frac{18679f_{n+4}}{60480} - \frac{281f_{n+5}}{60480} \\
h^4 y_{n+3}^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5 \left(\frac{-1810682f_{n+2}}{60480} - \frac{3887640f_{n+3}}{60480} + \frac{13676640f_{n+4}}{60480} - \frac{181f_{n+5}}{60480} \right) \quad \square \\
&\quad + \frac{1810682f_{n+2}}{60480} - \frac{3887640f_{n+3}}{60480} + \frac{13676640f_{n+4}}{60480} - \frac{181f_{n+5}}{60480} \\
h^4 y_{n+4}^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5 \left(\frac{-17417f_{n+2}}{60480} - \frac{2591f_{n+3}}{60480} + \frac{2539f_{n+4}}{60480} - \frac{281f_{n+5}}{60480} \right) \quad \square \\
&\quad + \frac{17417f_{n+2}}{60480} + \frac{2591f_{n+3}}{60480} - \frac{2539f_{n+4}}{60480} + \frac{281f_{n+5}}{60480} \\
h^4 y_{n+5}^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5 \left(\frac{12329f_{n+2}}{60480} + \frac{34417f_{n+3}}{60480} + \frac{4843f_{n+4}}{60480} - \frac{853f_{n+5}}{60480} \right) \quad \square \\
&\quad + \frac{12329f_{n+2}}{60480} + \frac{34417f_{n+3}}{60480} + \frac{4843f_{n+4}}{60480} - \frac{853f_{n+5}}{60480} \\
&\quad + \frac{22451f_{n+2}}{60480} + \frac{17659f_{n+3}}{60480} + \frac{84149f_{n+4}}{60480} + \frac{19097f_{n+5}}{60480} \quad \square
\end{aligned} \tag{3.2.30}$$

where $n = 0, 5, \dots, N - 5$. The formulae in (3.2.24), (3.2.26), (3.2.28) and (3.2.30) are the additional methods.

Remark 3.2.4. The formulae (3.2.22), (3.2.24), (3.2.26), (3.2.28) and (3.2.30) together form the Block Unification Method which is used to solve (2.1.1) for $\mu = 5$ numerically with appropriate boundary conditions.

3.2.6 Development of the Boundary Value Method (BVM)

Similar to the derivation of BVM for $\mu = 4$, (3.2.20) is evaluated at different points.

By evaluating $p(x)$, $p'(x)$, $p''(x)$, $p'''(x)$ and $p^{(iv)}(x)$, in (3.2.20) at the points x_{n+5} which implies $t = 1$, the following discrete 5-step LMMs are obtained:

$$\begin{aligned}
y_{n+5} &= -y_n + 5y_{n+1} - 10y_{n+2} + 10y_{n+3} - 5y_{n+4} + \frac{1}{24} h^5 (f_{n+1} \\
&\quad + 11f_{n+2} + 11f_{n+3} + f_{n+4}) \\
hy_{+5}^I &= \frac{25y_n}{6} - \frac{61y_{n+1}}{12} + \frac{39y_{n+2}}{12} - \frac{107y_{n+3}}{12} + \frac{77y_{n+4}}{10080} - \frac{h^5}{10080} (3f_n - 894f_{n+1} \\
&\quad + 9679f_{n+2} - 10819f_{n+3} + 1624f_{n+4} - 3f_{n+5}) (161f_n - 37795f_{n+1} \\
&\quad - 412860f_{n+2} - 535040f_{n+3} - 145805f_{n+4} - 2661f_{n+5}) \\
h^3 y_{+5}^{III} &= \frac{5y_n^2}{12} - 11y_{n+1} + 18y_{n+2} - 13y_{n+3} + \frac{7y_{n+4}^2}{60480} + \frac{60480}{h^5} (170f_n + 5239f_{n+1} \\
&\quad + 75876f_{n+2} + 108166f_{n+3} + 63434f_{n+4} + 4155f_{n+5}) \\
h^4 y_{+5}^{(iv)} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + \frac{h^5}{60480} (853f_n - 2879f_{n+1} \\
&\quad + 44902f_{n+2} + 35318f_{n+3} + 84149f_{n+4} + 19097f_{n+5})
\end{aligned} \tag{3.2.31}$$

where $n = 0, 5, \dots, N - 5$.

For the additional method, by evaluating $p'(x)$ in (3.2.20) at the points $x = x_i$, $i = 0(1)4$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}$, the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
hy_0^I &= -\frac{25y_0}{12} + 4y_1 - 3y_2 + \frac{4y_3}{3} - \frac{y_4}{4} + \frac{h^5}{10080} (3f_0 + 749f_1 + 1194f_2 + 54f_3 \\
&\quad + 19f_4 - 3f_5) \\
hy_1^I &= -\frac{y_0}{12} - \frac{5y_1}{6} + \frac{3y_2}{2} - \frac{y_3}{2} + \frac{y_4}{12} + \frac{h^5}{10080} (3f_0 - 120f_1 - 361f_2 - 21f_3 \\
&\quad - 6f_4 + f_5) \\
hy_2^I &= \frac{y_0}{12} - \frac{2y_1}{3} + \frac{2y_2}{3} - \frac{y_3}{12} - \frac{h^5}{10080} (f_0 - 44f_1 - 250f_2 - 44f_3 + f_4) \\
hy_3^I &= -\frac{y_0}{12} + y_1 - \frac{3y_2}{2} + \frac{5y_3}{6} + \frac{y_4}{4} - \frac{h^5}{10080} (36f_1 + 341f_2 + 135f_3 - 9f_4 + f_5) \\
&\quad - \frac{y_0}{12} - \frac{y_1}{2} + \frac{y_2}{2} - \frac{y_3}{6} + \frac{y_4}{4} - \frac{h^5}{10080}
\end{aligned}$$

$$hy_4^{\downarrow} = y_0 - 4y_1 + 3y_2 - 4y_3 + \underline{25y_4} + h^5 \quad \square$$

$$(f_0 + 99f_1 + 1134f_2 + 794f_3$$

[

(3.2.32)

$$-15f_4 + 3f_5)$$

By evaluating $p^{\mathfrak{u}}(x)$ in (3.2.20) at the points $x = x_i$, $i = 0(1)4$, which implies $t =$

$-1, -5, -5, 5, 5$, the following 5-step discrete LMMs are also obtained:

$$\begin{aligned}
h^2 y_0^{\text{II}} &= \frac{35y_0}{12} - \frac{26y_1}{3} + \frac{19y_2}{2} - \frac{14y_3}{3} + \frac{11y_4}{12} - \frac{h^5}{302400} (2661f_0 + 109055f_1 + 130790f_2 \\
&\quad + 8610f_3 + 1045f_4 - 161f_5) \quad \square \\
h^2 y_1^{\text{II}} &= \frac{11y_0}{12} - \frac{5y_1}{3} + \frac{y_2}{2} + \frac{y_3}{3} - \frac{y_4}{12} - \frac{h^5}{302400} (161f_0 - 9855f_1 - 15580f_2 + 520f_3) \quad \square \\
h^2 y_2^{\text{II}} &= \frac{-y_0 + 4y_1 - y_2 + 4y_3 - y_4}{12} + \frac{h^5}{302400} (79f_0 - 1685f_1 - 510f_2 + 2450f_3) \quad \square \\
h^2 y_3^{\text{II}} &= \frac{-y_0 + y_1 + y_2 - 5y_3 + 11y_4}{12} - \frac{h^5}{302400} (51f_0 + 665f_1 + 14000f_2 + 11040f_3) \quad \square \\
h^2 y_4^{\text{II}} &= \frac{-y_0 - \frac{14y_1}{3} + \frac{19y_2}{2} - \frac{26y_3}{3} + \frac{35y_4}{12}}{12} + \frac{h^5}{302400} (79f_0 + 11025f_1 + 127570f_2 \\
&\quad + 111470f_3 + 1695f_4 + 161f_5) \quad \square
\end{aligned} \tag{3.2.33}$$

By evaluating $p^{\text{II}}(x)$ in (3.2.20) at the points $x = x_i$, $i = 0(1)4$, which implies $t = -1, -5, -5, 5, 5$, the following 5-step discrete LMMs are also obtained:

$$\begin{aligned}
h^3 y_0^{\text{III}} &= -\frac{5y_0}{2} + 9y_1 - 12y_2 + 7y_3 - \frac{3y_4}{2} + \frac{h^5}{60480} (4155f_0 + 57134f_1 + 38866f_2 \\
&\quad + 6576f_3 - 1061f_4 + 170f_5) \quad \square \\
h^3 y_1^{\text{III}} &= -\frac{3y_0}{2} + 5y_1 - 6y_2 + 3y_3 - \frac{y_4}{2} - \frac{h^5}{60480} (170f_0 - 1395f_1 - 13004f_2 - 686f_3 \\
&\quad - 246f_4 + 41f_5) \quad \square \\
h^3 y_2^{\text{III}} &= \frac{-y_0}{2} + y_1 - y_3 + \frac{y_4}{2} + \frac{60480}{2} (41f_0 - 1676f_1 - 11850f_2 - 1676f_3 + 41f_4) \quad \square \\
h^3 y_3^{\text{III}} &= \frac{y_0}{2} - 3y_1 + 6y_2 - 5y_3 + \frac{3y_4}{2} + \frac{60480}{2} (1301f_1 + 12184f_2 + 2010f_3 \\
&\quad - 416f_4 + 41f_5) \quad \square \\
h^3 y_4^{\text{III}} &= \frac{3y_0}{2} - 7y_1 + 12y_2 - 9y_3 + \frac{5y_4}{2} - \frac{h^5}{60480} (41f_0 - 4026f_1 - 42266f_2 - 54584f_3 \\
&\quad - 5175f_4 + 170f_5) \quad \square
\end{aligned} \tag{3.2.34}$$

Finally, by evaluating $p^{(\text{IV})}(x)$ in (3.2.20) at the point $x = x_i$, $i = 0(1)4$, which implies $t = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}$, the following 5-step discrete LMMs are obtained:

$$\begin{aligned}
h^4 y_0^{(iv)} &= y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^5}{60480} (19097f_0 + 81629f_1 + 7598f_2 \\
&\quad + 17182f_3 - 5399f_4 + 853f_5) \quad \square \\
h^4 y_1^{(iv)} &= y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 + \frac{60480}{h^5} (853f_0 - 21695f_1 - 41114f_2 \\
&\quad + 3062f_3 - 1867f_4 + 281f_5) \quad \square \\
h^4 y_2^{(iv)} &= y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^5}{60480} (281f_0 - 5059f_1 - 1810f_2 \\
&\quad + 7774f_3 - 1367f_4 + 181f_5) \quad \square \\
h^4 y_3^{(iv)} &= y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 + \frac{60480}{h^5} (181f_0 + 1153f_1 + 35494f_2 \\
&\quad + 25910f_3 - 2539f_4 + 281f_5) \quad \square \\
h^4 y_4^{(iv)} &= y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^5}{60480} (281f_0 - 4387f_1 - 24658f_2 \\
&\quad - 68834f_3 - 24215f_4 + 853f_5) \quad \square
\end{aligned} \tag{3.2.35}$$

The formulae in (3.2.32)-(3.2.35) together form additional methods

Remark 3.2.5. The formulae (3.2.31)-(3.2.35) together form the Boundary Value Method (BVM) which is used to solve (2.1.1) for $\mu = 5$ numerically with appropriate boundary conditions.

Remark 3.2.6. For both Block Unification Method and the Boundary Value Method, If IVP is to be solved with these methods, it would be applied sequentially on block of intervals of the form $[x_n, x_{n+5}]$, $n = 0, 5, \dots, N - 5$, where N , the number of subintervals, must be a multiple of 5 in order to reach the final point $x_N = b$. In case of a BVP, all the formulae in each case of the BUM and BVM respectively are considered for $n = 0, 5, \dots, N - 5$. This results in a system of $5N$ equations, which together with the five boundary conditions leads to a system of $5N + 5$ equations in $5N + 5$ unknowns $\{y_j, y_j', y_j'', y_j''', y_j^{(iv)}\}$ for $j = 0(1)N$.

Remark 3.2.7. The BUM and BVM are classes of methods which are equally weighted for the numerical integration of both initial and boundary value problems.

3.2.7 Method of Lines (MOLs)

In order to apply BUM and the BVM to PDEs, a problem is first converted to system of ODEs via the method of lines. The method of lines approach is traditionally used for solving PDEs, whereby the a PDE is converted into a system of ODEs, by replacing the derivatives with their finite difference approximations (Ramos and Vigo-Aguiar, 2007; Brugnano and Trigliante, 1998). Following Jator (2015) and Biala *et al.* (2015), we demonstrate the process of converting (2.1.2) and (2.1.3), respectively, into a system of ODEs and then solve by the BUM and the BVM. Thus, for real numbers L_1, L_2, L_3, L_4 , solution $y(x, t)$ of (2.1.2), where (x, t) is in the rectangle $[L_1, L_2] \times [L_3, L_4]$. The t variable is discretised with mesh spacing $\Delta t = \frac{L_4 - L_3}{M}$, $t_m = L_3 + m\Delta t$, $m = 0, 1, \dots, M$, $\Delta x = \frac{L_2 - L_1}{N}$, $x_n = L_1 + n\Delta x$, $n = 0, 1, \dots, N$ with the vector

$$\mathbf{y} = [y_{1,1}, y_{1,2}, y_{2,1}, \dots, y_{n-1,m-1}]^T$$

and

$$\mathbf{G} = [G_{1,1}, G_{1,2}, G_{2,1}, \dots, G_{n-1,m-1}]^T$$

where $y_m \approx y(x, t_m)$ and $G_m \approx G(x, t_m)$;

3.2.7.1 Central Difference Approximations

If a function $u(x)$ can be evaluated at values that lie to the left and right of x on the real line, then the best two-point formulae will involve abscissae that are chosen symmetrically on both sides of x .

Theorem 3.2.1 (Centered Formula of Order $O(h^2)$). *[John and Kurtis (2004)] Assume that $u \in C^3[a, b]$ and that $x - h, x, x + h \in [a, b]$. Then*

$$u(x) \approx \frac{u_{n+1} - u_{n-1}}{2h}, \quad (3.2.36)$$

Furthermore, there exists a number $\xi = \xi(x) \in [a, b]$ such that

$$u^j(x) = \frac{u_{n+1} - u_{n-1}}{2h} + T(u, h), \quad (3.2.37)$$

where

$$T(u, h) = -\frac{h^2 u^{jj}(\xi)}{6} = O(h^2). \quad (3.2.38)$$

The term $T(u, h)$ is called the truncation error, $u_{n+1} = u(x_n + h)$, $u_{n-1} = u(x_n - h)$.

By similar derivation of (3.2.36), $u^j(x)$ has the formulae

$$u^j(x) \approx \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} \quad (3.2.39)$$

For PDE, where $u = u_{n,m} = u(x_n, t_m)$, and by discretising the space variable x , the central difference method becomes

$$\frac{\partial u}{\partial t} = u_t(x, t_m) \approx \frac{u(x, t_{m+1}) - u(x, t_{m-1}))}{2h}, \quad (3.2.40)$$

$$\frac{\partial^2 u}{\partial t^2} = u_{tt}(x, t_m) \approx \frac{u(x, t_{m+1}) - 2u(x, t_m) + u(x, t_{m-1}))}{h^2}, \quad (3.2.41)$$

The formulae (3.2.40) and (3.2.41) are called semi-discretisation of the dependent variable u .

Using the central difference approximation (3.2.41)

$$y_{tt}(x, t_m) \approx \frac{y(x, t_{m+1}) - 2y(x, t_m) + y(x, t_{m-1}))}{(\Delta t)^2}.$$

then (2.1.2) has the following semidiscretized form

$$\frac{dy_m^4}{dx^4} = - \frac{y_{m+1}}{4} - \frac{2y_m + y_{m-1}}{(\Delta t)^2} + G_m \quad (3.2.42)$$

which can be written in the form

$$y^{(5)} = f(x, y) \quad (3.2.43)$$

subject to the appropriate initial and boundary conditions, where $f(x, y) = Ay + g$, and A is an $(M-1) \times (M-1)$ matrix arising from the semi-discretized system (3.2.42).

Similarly (2.1.3) has the semidiscretized form

$$\frac{dy^n}{dx^n} \approx - \frac{a_{n,m}}{b} + b_{n,m} \frac{1}{(\Delta t)^2} \frac{y_{m+1} - 2y_m + y_{m-1}}{\Sigma} + G_{n,m} \quad (3.2.44)$$

where

$$G_{n,m} = x_n, t_m, y_{n,m}, y_{n,m}^2, \frac{dy_{n,m}}{dx}, \frac{y_{m+1} - y_{m-1}}{(2\Delta t)}, \frac{dy_{n,m}^2}{dx^2}, \frac{dy_{n,m}^3}{dx^3} \Sigma$$

and $b_{n,m} = 0$ for $n = 0, 1, \dots, N$ and $m = 0, 1, \dots, M$. This forms a fifth-order problem in ODE of the type:

$$y^{(5)} = f(x, y, y^I, y^{II}, y^{III}, y^{(iv)}), \quad a < x < b \quad (3.2.45)$$

subject to the boundary conditions:

$$\begin{aligned} y(a) &= y_0, \quad y^I(a) = y_0^I, \quad y^{II}(a) = y_0^{II}, \\ y(b) &= y_{n,m}^I, \quad y^I(b) = y_{n,m}^2 \end{aligned}$$

where $f(x, y, y^I, y^{II}, y^{III}, y^{(iv)}) = AY + g$ and A is a $k \times k$ matrix ($k = (N-1)(M-1)$) arising from the semidiscretized system (2.1.3) and which is expressed in the form (3.2.45). Here g is a vector of constants. Note that j in $y_{n,m}^j$, for $j = 1, 2$, are only index notation for different $y_{n,m}$.

The equivalent form of the BUM (3.2.7), (3.2.9), (3.2.11) and (3.2.13) for the direct numerical integration of problem (2.1.2) is given as

$$\begin{aligned}
y_{n+4,m} &= 4y_{n+3,m} - 6y_{n+2,m} + 4y_{n+1,m} - y_{n,m} + h^4 \left[-\frac{1}{720}f_{n,m} + \frac{31}{180}f_{n+1,m} \right. \\
&\quad \left. + \frac{79}{120}f_{n+2,m} + \frac{31}{180}f_{n+3,m} - \frac{1}{720}f_{n+4,m} \right] \\
y_{n+5,m} &= -4y_{n,m} + 15y_{n+1,m} - 20y_{n+2,m} + 10y_{n+3,m} + h^4 \left[-\frac{1}{180}f_{n,m} + \frac{11}{16}f_{n+1,m} \right. \\
&\quad \left. + \frac{101}{36}f_{n+2,m} + \frac{97}{72}f_{n+3,m} + \frac{1}{6}f_{n+4,m} - \frac{1}{720}f_{n+5,m} \right] \\
hy_{n,m}^\mu &= -\frac{11y_{n,m}}{6} + 3y_{n+1,m} - \frac{3y_{n+2,m}}{2} + \frac{y_{n+3,m}}{3} - h^4 \left[-\frac{937}{100800}f_{n,m} - \frac{19}{105}f_{n+1,m} \right. \\
&\quad \left. - \frac{599}{10080}f_{n+2,m} - \frac{1}{720}f_{n+3,m} + \frac{3}{2240}f_{n+4,m} - \frac{1}{3600}f_{n+5,m} \right] \\
hy_{n+1,m}^\mu &= -y_{n,m} - \frac{1}{2}y_{n+1,m} + y_{n+2,m} - \frac{1}{6}y_{n+3,m} + h^4 \left[-\frac{1}{5400}f_{n,m} \right. \\
&\quad \left. + \frac{2809}{60480}f_{n+1,m} + \frac{43}{1008}f_{n+2,m} - \frac{229}{30240}f_{n+3,m} + \frac{1}{432}f_{n+4,m} - \frac{11}{33600}f_{n+5,m} \right] \\
hy_{n+2,m}^\mu &= \frac{1}{6}y_{n,m} - y_{n+1,m} + \frac{1}{2}y_{n+2,m} + \frac{1}{3}y_{n+3,m} + h^4 \left[-\frac{169}{302400}f_{n,m} \right. \\
&\quad \left. - \frac{311}{10080}f_{n+1,m} - \frac{353}{6048}f_{n+2,m} + \frac{1}{135}f_{n+3,m} - \frac{7}{2880}f_{n+4,m} + \frac{53}{151200}f_{n+5,m} \right] \\
hy_{n+3,m}^\mu &= -\frac{1}{3}y_{n,m} + \frac{3}{2}y_{n+1,m} - 3y_{n+2,m} + \frac{11}{6}y_{n+3,m} + h^4 \left[-\frac{41}{50400}f_{n,m} \right. \\
&\quad \left. + \frac{173}{2880}f_{n+1,m} + \frac{11}{60}f_{n+2,m} + \frac{61}{10080}f_{n+3,m} + \frac{17}{10080}f_{n+4,m} - \frac{11}{33600}f_{n+5,m} \right] \\
hy_{n+4,m}^\mu &= -\frac{11}{6}y_{n,m} + 7y_{n+1,m} - \frac{19}{2}y_{n+2,m} + \frac{13}{3}y_{n+3,m} + h^4 \left[-\frac{671}{302400}f_{n,m} \right. \\
&\quad \left. + \frac{169}{540}f_{n+1,m} + \frac{12821}{10080}f_{n+2,m} + \frac{7447}{15120}f_{n+3,m} + \frac{509}{60480}f_{n+4,m} - \frac{1}{3600}f_{n+5,m} \right] \\
hy_{n+5,m}^\mu &= -\frac{13}{3}y_{n,m} + \frac{31}{2}y_{n+1,m} - 19y_{n+2,m} + \frac{47}{6}y_{n+3,m} + h^4 \left[-\frac{31}{5400}f_{n,m} \right. \\
&\quad \left. + \frac{1663}{2240}f_{n+1,m} + \frac{6847}{2160}f_{n+2,m} + \frac{60863}{30240}f_{n+3,m} + \frac{2473}{5040}f_{n+4,m} + \frac{2041}{302400}f_{n+5,m} \right]
\end{aligned} \tag{3.2.46}$$

$$\begin{aligned}
h^2 y_{n,m}^{\text{II}} &= 2y_{n,m} - 5y_{n+1,m} + 4y_{n+2,m} - y_{n+3,m} + h^2 \left[-\frac{1411}{20160} f_{n,m} + \frac{3091}{4320} f_{n+1,m} \right. \\
&\quad \left. + \frac{2831}{30240} f_{n+2,m} + \frac{143}{2520} f_{n+3,m} - \frac{1391}{60480} f_{n+4,m} + \frac{23}{6048} f_{n+5,m} \right] \\
h^2 y_{n+1,m}^{\text{II}} &= y_{n,m} - 2y_{n+1,m} + y_{n+2,m} + h^4 \left[-\frac{73}{30240} f_{n,m} - \frac{1601}{20160} f_{n+1,m} \right. \\
&\quad \left. + \frac{1}{7560} f_{n+2,m} - \frac{11}{4320} f_{n+3,m} + \frac{11}{10080} f_{n+4,m} - \frac{11}{60480} f_{n+5,m} \right] \\
h^2 y_{n+2,m}^{\text{II}} &= y_{n+1,m} - 2y_{n+2,m} + y_{n+3,m} + h^4 \left[-\frac{11}{60480} f_{n,m} - \frac{53}{15120} f_{n+1,m} \right. \\
&\quad \left. - \frac{773}{10080} f_{n+2,m} - \frac{53}{15120} f_{n+3,m} + \frac{11}{60480} f_{n+4,m} \right] \\
h^2 y_{n+3,m}^{\text{II}} &= -y_{n,m} + 4y_{n+1,m} - 5y_{n+2,m} + 2y_{n+3,m} + h^4 \left[-\frac{1}{220} f_{n,m} + \frac{10427}{60480} f_{n+1,m} \right. \\
&\quad \left. + \frac{9901}{15120} f_{n+2,m} + \frac{107}{1120} f_{n+3,m} - \frac{37}{7560} f_{n+4,m} + \frac{11}{60480} f_{n+5,m} \right] \\
h^2 y_{n+4,m}^{\text{II}} &= -2y_{n,m} + 7y_{n+1,m} - 8y_{n+2,m} + 3y_{n+3,m} + h^4 \left[-\frac{179}{60480} f_{n,m} + \frac{3469}{10080} f_{n+1,m} \right. \\
&\quad \left. + \frac{6421}{4320} f_{n+2,m} + \frac{3791}{3780} f_{n+3,m} + \frac{121}{1344} f_{n+4,m} - \frac{23}{6048} f_{n+5,m} \right] \\
h^2 y_{n+5,m}^{\text{II}} &= -3y_{n,m} + 10y_{n+1,m} - 11y_{n+2,m} + 4y_{n+3,m} + h^4 \left[-\frac{11}{30240} f_{n,m} + \frac{29689}{60480} f_{n+1,m} \right. \\
&\quad \left. + \frac{499}{210} f_{n+2,m} + \frac{58271}{30240} f_{n+3,m} + \frac{4561}{4320} f_{n+4,m} + \frac{271}{4032} f_{n+5,m} \right] \\
h^3 y_{n,m}^{\text{III}} &= -y_{n,m} + 3y_{n+1,m} - 3y_{n+2,m} + y_{n+3,m} - h^3 \left[-\frac{19151}{60480} f_{n,m} - \frac{73967}{60480} f_{n+1,m} \right. \\
&\quad \left. + \frac{1261}{6048} f_{n+2,m} - \frac{7439}{30240} f_{n+3,m} + \frac{5549}{60480} f_{n+4,m} - \frac{883}{60480} f_{n+5,m} \right] \\
h^3 y_{n+1,m}^{\text{III}} &= -y_{n,m} + 3y_{n+1,m} - 3y_{n+2,m} + y_{n+3,m} + h^4 \left[-\frac{799}{60480} f_{n,m} - \frac{14033}{60480} f_{n+1,m} \right. \\
&\quad \left. - \frac{10453}{30240} f_{n+2,m} + \frac{2683}{30240} f_{n+3,m} - \frac{1717}{60480} f_{n+4,m} + \frac{251}{60480} f_{n+5,m} \right] \\
h^3 y_{n+2,m}^{\text{III}} &= -y_{n,m} + 3y_{n+1,m} - 3y_{n+2,m} + y_{n+3,m} + h^4 \left[-\frac{67}{12096} f_{n,m} + \frac{12721}{60480} f_{n+1,m} \right. \\
&\quad \left. + \frac{11009}{30240} f_{n+2,m} - \frac{547}{6048} f_{n+3,m} + \frac{1517}{60480} f_{n+4,m} - \frac{211}{60480} f_{n+5,m} \right] \\
h^3 y_{n+3,m}^{\text{III}} &= -y_{n,m} + 3y_{n+1,m} - 3y_{n+2,m} + y_{n+3,m} + h^4 \left[-\frac{127}{60480} f_{n,m} + \frac{1763}{12096} f_{n+1,m} \right. \\
&\quad \left. + \frac{27851}{30240} f_{n+2,m} + \frac{14107}{30240} f_{n+3,m} - \frac{2389}{60480} f_{n+4,m} + \frac{251}{60480} f_{n+5,m} \right] \\
h^3 y_{n+4,m}^{\text{III}} &= -y_{n,m} + 3y_{n+1,m} - 3y_{n+2,m} + y_{n+3,m} + h^4 \left[-\frac{67}{12096} f_{n,m} + \frac{12049}{60480} f_{n+1,m} \right. \\
&\quad \left. + \frac{22433}{30240} f_{n+2,m} + \frac{35569}{30240} f_{n+3,m} + \frac{4873}{12096} f_{n+4,m} - \frac{883}{60480} f_{n+5,m} \right] \\
h^3 y_{n+5,m}^{\text{III}} &= -y_{n,m} + 3y_{n+1,m} - 3y_{n+2,m} + y_{n+3,m} + h^4 \left[-\frac{799}{60480} f_{n,m} + \frac{4783}{60480} f_{n+1,m} \right. \\
&\quad \left. + \frac{6511}{6048} f_{n+2,m} + \frac{18811}{30240} f_{n+3,m} + \frac{84299}{60480} f_{n+4,m} + \frac{19067}{60480} f_{n+5,m} \right]
\end{aligned}
\tag{3.2.47}$$

Similarly, the equivalent form of the BUM (3.2.22), (3.2.24), (3.2.26), (3.2.28) and (3.2.30) for the direct numerical integration of problem (2.1.3) is given as

$$\begin{aligned}
y_{n+5,m} &= y_{n,m} - 5y_{n+1,m} + 10y_{n+2,m} - 10y_{n+3,m} + 5y_{n+4,m} + h^5 \cdot \frac{1}{24} f_{n+1,m} \\
&\quad + \frac{11}{24} h^5 f_{n+2,m} + \frac{11}{24} h^5 f_{n+3,m} + \frac{1}{24} h^5 f_{n+4,m} \\
hy_{n,m} &= -\frac{25y_{n,m}}{12} - 4y_{n+1,m} - 3y_{n+2,m} + \frac{4y_{n+3,m}}{3} - \frac{y_{n+4,m}}{4} + h^5 \cdot \frac{f_{n,m}}{3360} + \frac{107f_{n+1,m}}{1440} \\
&\quad + \frac{199f_{n+2,m}}{1680} + \frac{3f_{n+3,m}}{560} + \frac{19f_{n+4,m}}{10080} - \frac{f_{n+5,m}}{3360} \\
hy_{n+1,m} &= -\frac{y_{n,m}}{4} - \frac{5y_{n+1,m}}{6} + \frac{3y_{n+2,m}}{2} - \frac{y_{n+3,m}}{2} + \frac{y_{n+4,m}}{12} + h^5 \cdot \frac{f_{n,m}}{3360} - \frac{f_{n+1,m}}{84} \\
&\quad - \frac{361f_{n+2,m}}{10080} - \frac{f_{n+3,m}}{480} - \frac{f_{n+4,m}}{1680} + \frac{f_{n+5,m}}{10080} \\
hy_{n+2,m} &= \frac{y_{n,m}}{12} - \frac{2y_{n+1,m}}{3} + \frac{2y_{n+3,m}}{3} - \frac{y_{n+4,m}}{12} + h^5 \cdot \frac{-f_{n,m}}{10080} + \frac{11f_{n+1,m}}{2520} \\
&\quad + \frac{25f_{n+2,m}}{1008} + \frac{11f_{n+3,m}}{2520} - \frac{f_{n+4,m}}{10080} \\
hy_{n+3,m} &= -\frac{y_{n,m}}{12} + \frac{y_{n+1,m}}{2} - \frac{3y_{n+2,m}}{2} + \frac{5y_{n+3,m}}{6} + \frac{y_{n+4,m}}{4} - h^5 \cdot \frac{f_{n+1,m}}{280} \\
&\quad - \frac{341f_{n+2,m}}{10080} - \frac{3f_{n+3,m}}{224} + \frac{f_{n+4,m}}{1120} - \frac{f_{n+5,m}}{1008} \\
hy_{n+4,m} &= \frac{y_{n,m}}{4} - \frac{4y_{n+1,m}}{3} + 3y_{n+2,m} - 4y_{n+3,m} + \frac{25y_{n+4,m}}{12} + h^5 \cdot \frac{f_{n,m}}{10080} + \frac{11f_{n+1,m}}{1120} \\
&\quad + \frac{9f_{n+2,m}}{80} + \frac{397f_{n+3,m}}{5040} - \frac{f_{n+4,m}}{672} + \frac{f_{n+5,m}}{3360} \\
hy_{n+5,m} &= \frac{25y_{n,m}}{12} - \frac{61y_{n+1,m}}{6} + \frac{39y_{n+2,m}}{2} - \frac{107y_{n+3,m}}{6} + \frac{77y_{n+4,m}}{12} + h^5 \cdot \frac{-f_{n,m}}{3360} + \frac{149f_{n+1,m}}{1680} \\
&\quad + \frac{9679f_{n+2,m}}{10080} + \frac{10819f_{n+3,m}}{10080} + \frac{29f_{n+4,m}}{180} + \frac{f_{n+5,m}}{3360}
\end{aligned}$$

(3.2.48)

$$\begin{aligned}
h^2 y_{n,m}^{\text{II}} &= \frac{35y_{n,m}}{12} - \frac{26y_{n+1,m}}{3} + \frac{19}{2}y_{n+2,m} - \frac{14y_{n+3,m}}{3} + \frac{11y_{n+4,m}}{12} + h^5 \left[\frac{-887f_{n,m}}{100800} - \frac{21811f_{n+1,m}}{60480} \right. \\
&\quad \left. - \frac{13079f_{n+2,m}}{30240} - \frac{41f_{n+3,m}}{1440} - \frac{209f_{n+4,m}}{60480} + \frac{23f_{n+5,m}}{43200} \right] \\
h^2 y_{n+1,m}^{\text{II}} &= \frac{11y_{n,m}}{12} - \frac{5y_{n+1,m}}{3} + \frac{y_{n+2,m}}{2} + \frac{y_{n+3,m}}{3} - \frac{y_{n+4,m}}{12} + h^5 \left[\frac{-23h^3 f_{n,m}}{43200} + \frac{73f_{n+1,m}}{2240} \right. \\
&\quad \left. + \frac{779f_{n+2,m}}{15120} - \frac{13f_{n+3,m}}{7560} + \frac{f_{n+4,m}}{576} - \frac{79f_{n+5,m}}{302400} \right] \\
h^2 y_{n+2,m}^{\text{II}} &= -\frac{y_{n,m}}{12} + \frac{4y_{n+1,m}}{3} - \frac{5y_{n+2,m}}{2} + \frac{4y_{n+3,m}}{3} - \frac{y_{n+4,m}}{12} + h^5 \left[\frac{79f_{n,m}}{302400} - \frac{337f_{n+1,m}}{60480} \right. \\
&\quad \left. - \frac{17f_{n+2,m}}{10080} + \frac{7f_{n+3,m}}{864} - \frac{11f_{n+4,m}}{8640} + \frac{17f_{n+5,m}}{100800} \right] \\
h^2 y_{n+3,m}^{\text{II}} &= -\frac{y_{n,m}}{12} + \frac{y_{n+1,m}}{3} + \frac{y_{n+2,m}}{2} - \frac{5y_{n+3,m}}{3} + \frac{11y_{n+4,m}}{12} + h^5 \left[\frac{-17f_{n,m}}{100800} - \frac{19f_{n+1,m}}{8640} \right. \\
&\quad \left. - \frac{5f_{n+2,m}}{108} - \frac{23f_{n+3,m}}{630} + \frac{127f_{n+4,m}}{60480} - \frac{79f_{n+5,m}}{302400} \right] \\
h^2 y_{n+4,m}^{\text{II}} &= \frac{11y_{n,m}}{12} - \frac{14y_{n+1,m}}{3} + \frac{19y_{n+2,m}}{2} - \frac{26y_{n+3,m}}{3} + \frac{35y_{n+4,m}}{12} + h^5 \left[\frac{79f_{n,m}}{302400} + \frac{7f_{n+1,m}}{192} \right. \\
&\quad \left. + \frac{12757f_{n+2,m}}{30240} + \frac{11147f_{n+3,m}}{30240} + \frac{113f_{n+4,m}}{20160} + \frac{23f_{n+5,m}}{43200} \right] \\
h^2 y_{n+5,m}^{\text{II}} &= \frac{35y_{n,m}}{12} - \frac{41y_{n+1,m}}{3} + \frac{49y_{n+2,m}}{2} - \frac{59y_{n+3,m}}{3} + \frac{71y_{n+4,m}}{12} + h^5 \left[\frac{-23f_{n,m}}{43200} + \frac{7559f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{983f_{n+2,m}}{720} + \frac{1672f_{n+3,m}}{945} + \frac{29161f_{n+4,m}}{60480} + \frac{887f_{n+5,m}}{100800} \right]
\end{aligned}
\tag{3.2.49}$$

$$\begin{aligned}
h^3 y_{n,m}^{\text{III}} &= -\frac{5y_{n,m}}{2} + 9y_{n+1,m} - 12y_{n+2,m} + 7y_{n+3,m} - \frac{3y_{n+4,m}}{2} + h^5 \left[\frac{277f_{n,m}}{4032} + \frac{4081f_{n+1,m}}{4320} \right. \\
&\quad \left. + \frac{19433f_{n+2,m}}{30240} + \frac{137f_{n+3,m}}{1260} - \frac{1061f_{n+4,m}}{60480} + \frac{17f_{n+5,m}}{6048} \right] \\
h^3 y_{n+1,m}^{\text{III}} &= -\frac{3y_{n,m}}{2} + 5y_{n+1,m} - 6y_{n+2,m} + 3y_{n+3,m} - \frac{y_{n+4,m}}{2} + h^5 \left[\frac{-17f_{n,m}}{6048} + \frac{31f_{n+1,m}}{1344} \right. \\
&\quad \left. + \frac{3251f_{n+2,m}}{15120} + \frac{49f_{n+3,m}}{4320} + \frac{41f_{n+4,m}}{10080} - \frac{41f_{n+5,m}}{60480} \right] \\
h^3 y_{n+2,m}^{\text{III}} &= -\frac{y_{n,m}}{2} + y_{n+1,m} - y_{n+3,m} + \frac{y_{n+4,m}}{2} + h^5 \left[\frac{41f_{n,m}}{60480} - \frac{419f_{n+1,m}}{15120} \right. \\
&\quad \left. - \frac{395f_{n+2,m}}{2016} - \frac{419f_{n+3,m}}{15120} + \frac{41f_{n+4,m}}{60480} \right] \\
h^3 y_{n+3,m}^{\text{III}} &= \frac{y_{n,m}}{2} - 3y_{n+1,m} + 6y_{n+2,m} - 5y_{n+3,m} + \frac{3y_{n+4,m}}{2} + h^5 \left[\frac{1301f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{1523f_{n+2,m}}{7560} + \frac{67f_{n+3,m}}{2016} - \frac{13f_{n+4,m}}{1890} + \frac{41f_{n+5,m}}{60480} \right] \\
h^3 y_{n+4,m}^{\text{III}} &= \frac{3y_{n,m}}{2} - 7y_{n+1,m} + 12y_{n+2,m} - 9y_{n+3,m} + \frac{5y_{n+4,m}}{2} + h^5 \left[\frac{-41f_{n,m}}{60480} + \frac{671f_{n+1,m}}{10080} \right. \\
&\quad \left. + \frac{3019f_{n+2,m}}{4320} + \frac{6823f_{n+3,m}}{7560} + \frac{115f_{n+4,m}}{1344} - \frac{17f_{n+5,m}}{6048} \right] \\
h^3 y_{n+5,m}^{\text{III}} &= \frac{5y_{n,m}}{2} - 11y_{n+1,m} + 18y_{n+2,m} - 13y_{n+3,m} + \frac{7y_{n+4,m}}{2} + h^5 \left[\frac{17f_{n,m}}{6048} + \frac{5239f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{6323f_{n+2,m}}{5040} + \frac{54083f_{n+3,m}}{30240} + \frac{4531f_{n+4,m}}{4320} + \frac{277f_{n+5,m}}{4032} \right]
\end{aligned}
\tag{3.2.50}$$

$$\begin{aligned}
h^4 y_{n,m}^{(iv)} &= y_{n,m} - 4y_{n+1,m} + 6y_{n+2,m} - 4y_{n+3,m} + y_{n+4,m} + h^5 \left[\frac{-19097f_{n,m}}{60480} - \frac{81629f_{n+1,m}}{60480} \right. \\
&\quad \left. - \frac{3799f_{n+2,m}}{30240} - \frac{8591f_{n+3,m}}{30240} + \frac{5399f_{n+4,m}}{60480} - \frac{853f_{n+5,m}}{60480} \right] \\
h^4 y_{n+1,m}^{(iv)} &= y_{n,m} - 4y_{n+1,m} + 6y_{n+2,m} - 4y_{n+3,m} + y_{n+4,m} + h^5 \left[\frac{853f_{n,m}}{60480} - \frac{4339f_{n+1,m}}{12096} \right. \\
&\quad \left. - \frac{20557f_{n+2,m}}{30240} + \frac{1531f_{n+3,m}}{30240} - \frac{1867f_{n+4,m}}{60480} + \frac{281f_{n+5,m}}{60480} \right] \\
h^4 y_{n+2,m}^{(iv)} &= y_{n,m} - 4y_{n+1,m} + 6y_{n+2,m} - 4y_{n+3,m} + y_{n+4,m} + h^5 \left[\frac{-281f_{n,m}}{60480} + \frac{5059f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{181f_{n+2,m}}{60480} - \frac{3887f_{n+3,m}}{30240} + \frac{1367f_{n+4,m}}{60480} - \frac{181f_{n+5,m}}{60480} \right] \\
h^4 y_{n+3,m}^{(iv)} &= y_{n,m} - 4y_{n+1,m} + 6y_{n+2,m} - 4y_{n+3,m} + y_{n+4,m} + h^5 \left[\frac{181f_{n,m}}{60480} + \frac{1153f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{17747f_{n+2,m}}{30240} + \frac{2591f_{n+3,m}}{60480} - \frac{2539f_{n+4,m}}{60480} + \frac{281f_{n+5,m}}{60480} \right] \\
h^4 y_{n+4,m}^{(iv)} &= y_{n,m} - 4y_{n+1,m} + 6y_{n+2,m} - 4y_{n+3,m} + y_{n+4,m} + h^5 \left[\frac{-281f_{n,m}}{60480} + \frac{4387f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{12329f_{n+2,m}}{30240} + \frac{34417f_{n+3,m}}{30240} + \frac{4843f_{n+4,m}}{12096} - \frac{853f_{n+5,m}}{60480} \right] \\
h^4 y_{n+5,m}^{(iv)} &= y_{n,m} - 4y_{n+1,m} + 6y_{n+2,m} - 4y_{n+3,m} + y_{n+4,m} + h^5 \left[\frac{853f_{n,m}}{60480} - \frac{2879f_{n+1,m}}{60480} \right. \\
&\quad \left. + \frac{22451f_{n+2,m}}{30240} + \frac{17659f_{n+3,m}}{30240} + \frac{84149f_{n+4,m}}{60480} + \frac{19097f_{n+5,m}}{60480} \right]
\end{aligned} \tag{3.2.51}$$

where $n = 0, 5, \dots, N - 5$.

3.3 Analysis of the Methods

Numerical methods for solution of ODEs have underlined basic properties viz-a-viz the order and error constant of the method, consistency and convergence. These properties are analyzed herein.

3.3.1 Order and Error Constants

3.3.1.1 Order of a Method

The formulae

$$y(x) = \sum_{j=0}^{k_1} a_j(x) y_{n+j} + h^\mu \sum_{j=0}^{k_2} \beta_j(x) f_{n+j} \tag{3.3.1}$$

is associated with the linear difference operator $L[y(x); h]$ defined by

$$\mathbb{L}[y(x); h] \equiv y(x+jh) - \sum_{j=0}^k a_j y(x+jh) + h^\mu \sum_{j=0}^k \beta_j y^{(\mu)}(x+jh) \quad (3.3.2)$$

where $k = 5$ for $\mu = 4, 5$.

Expanding (3.3.6) in Taylor series, the following linear combinations of C_i are obtained

$$\mathbb{L}[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + O(h^{p+1}) \quad (3.3.3)$$

where the constants

$$\begin{aligned} C_0 &= \sum_{j=0}^k a_j \\ C_1 &= \sum_{j=1}^k j a_j \\ C_2 &= \frac{1}{2!} \sum_{j=1}^k j^2 a_j - \sum_{j=0}^k \beta_j \\ &\vdots \\ C_p &= \frac{1}{p!} \sum_{j=1}^k j^p a_j - \sum_{j=1}^k j^{p-2} \beta_j \end{aligned}$$

Definition 3.3.1. The LMM (3.3.1) is of order p if $C_0 = C_1 = C_2 = \cdots = C_{p+\mu} = 0$, and $C_{p+\mu+1} \neq 0$ in which

$$\mathbb{L}[y(x); h] = C_{p+\mu+1} h^{p+\mu+1} y^{(p+\mu+1)}(x) + O(h^{p+\mu+2}) \quad (3.3.4)$$

3.3.1.2 Order of the Block

Let the implicit block method formed by (3.2.7), (3.2.11) and (3.2.12) be defined by

$$\bar{Y}_{n+j} = \sum_{j=0}^{k_1} A_j \bar{y}_{n+j} + h^\mu \sum_{j=0}^{k_2} B_j \bar{f}_{n+j} \quad (3.3.5)$$

where μ is the degree of the derivative, A_j and B_j are constant coefficients, k_1, k_2 are 3, 5 for $\mu = 4$ and k_1, k_2 are 4, 5 for $\mu = 5$, where

$$\begin{aligned}
 & \begin{pmatrix} y_{n+\mu} \\ hy_n^j \\ hy_{n+1}^j \\ \vdots \\ hy_{n+\mu}^j \\ h^2 y_n^j \\ h^2 y_{n+1}^j \end{pmatrix} = \begin{pmatrix} a_1^{0,0} & a_2^{0,0} & \dots & a_{k_1}^{0,0} \\ a_1^{1,1} & a_2^{1,1} & \dots & a_{k_1}^{1,1} \\ a_1^{2,1} & a_2^{2,1} & \dots & a_{k_1}^{2,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{\mu,1} & a_2^{\mu,1} & \dots & a_{k_1}^{\mu,1} \\ a_1^{1,2} & a_2^{1,2} & \dots & a_{k_1}^{1,2} \\ a_1^{2,2} & a_2^{2,2} & \dots & a_{k_1}^{2,2} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix} \\
 Y_{n+j} &= \begin{pmatrix} \vdots \\ h^2 y_{n+\mu}^j \\ \vdots \\ h^{\mu-1} y_{n+1}^{(\mu-1)} \\ h^{\mu-1} y_{n+1}^{(\mu-1)} \\ \vdots \\ h^{\mu-1} y_{n+1}^{(\mu-1)} \end{pmatrix} A \begin{pmatrix} \vdots \\ a_1^{\mu,2} & a_2^{\mu,2} & \dots & a_{k_1}^{\mu,2} \\ \vdots \\ a_1^{1,\mu} & a_2^{1,\mu} & \dots & a_{k_1}^{1,\mu} \\ a_1^{2,\mu} & a_2^{2,\mu} & \dots & a_{k_1}^{2,\mu} \\ \vdots \\ a_1^{\mu,\mu} & a_2^{\mu,\mu} & \dots & a_{k_1}^{\mu,\mu} \end{pmatrix} \begin{pmatrix} y_{n+j} \\ \vdots \\ y_{n+k_1} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
& \begin{bmatrix} \beta_1^{0,0} & \beta_2^{0,0} & \cdots & \beta_{k_2}^{0,0} \\ \beta_1^{1,1} & \beta_2^{1,1} & \cdots & \beta_{k_2}^{1,1} \\ \beta_1^{2,1} & \beta_2^{2,1} & \cdots & \beta_{k_2}^{2,1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{\mu,1} & \beta_2^{\mu,1} & \cdots & \beta_{k_2}^{\mu,1} \\ \beta_1^{1,2} & \beta_2^{1,2} & \cdots & \beta_{k_2}^{1,2} \\ \beta_1^{2,2} & \beta_2^{2,2} & \cdots & \beta_{k_2}^{2,2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{\mu,2} & \beta_2^{\mu,2} & \cdots & \beta_{k_2}^{\mu,2} \end{bmatrix} = \begin{bmatrix} f_{n+1}^n \\ f_{n+1}^{n+1} \\ f_{n+1}^{n+2} \\ \vdots \\ f_{n+1}^{n+\mu} \end{bmatrix} \\
& B = \begin{bmatrix} \beta_1^{\mu,2} & \beta_2^{\mu,2} & \cdots & \beta_{k_2}^{\mu,2} \\ \beta_1^{1,\mu} & \beta_2^{1,\mu} & \cdots & \beta_{k_2}^{1,\mu} \\ \beta_1^{2,\mu} & \beta_2^{2,\mu} & \cdots & \beta_{k_2}^{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{\mu,\mu} & \beta_2^{\mu,\mu} & \cdots & \beta_{k_2}^{\mu,\mu} \end{bmatrix}
\end{aligned}$$

The linear difference operator $L[\bar{y}(x); h]$ is defined by

$$L[\bar{y}(x); h] \equiv Y(x + jh) - \sum_{j=0}^{k_1} A_j y(x + jh) + h^\mu \sum_{j=0}^{k_2} B_j f(x + jh) \quad (3.3.6)$$

Expanding (3.3.6) in Taylor series, the following linear combinations of C_i are obtained

$$L[\bar{y}(x); h] = \bar{C}_0 y(x) + \bar{C}_1 h y'(x) + \bar{C}_2 h^2 y''(x) + \cdots + \bar{C}_p h^p y^{(p)}(x) + O(h^{p+1}) \quad (3.3.7)$$

where the constants $\bar{C}_j$'s are vectors.

Definition 3.3.2. The block method (3.3.5) is of order p if $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{p+\mu} = 0$, and $\bar{C}_{p+\mu+1} \neq 0$ in which

$$L[\bar{y}(x); h] = \bar{C}_{p+\mu+1} h^{p+\mu+1} y^{(p+\mu+1)}(x) + O(h^{p+\mu+2}) \quad (3.3.8)$$

In this case, $\bar{C}_{p+\mu+1}$ is vector of error constants.

Remark 3.3.1. It is remarked here that the formulae for the BUM and BVM are weighted equally; hence the local truncation error and order for either case are the same for each of $\mu = 4$ and $\mu = 5$.

Definition 3.3.3. The implicit method (3.3.1) is said to be consistent if the order $p > 1$.

Theorem 3.3.1. (Henrichi (1962)) The implicit method (3.3.1) is said to be consistent if it satisfies the following conditions:

$$1. \sum_{j=0}^k \alpha_j = 0$$

$$2. \rho(1) = \rho'(1) = 0 \text{ and}$$

$$3. \rho^{(\mu)}(1) = \mu! \sigma(1)$$

where μ is the order of the differential equation.

Remark 3.3.2. Definition 3.3.3 is a sufficient condition for the block method (3.3.5) to be consistent, $p \geq 1$, Jator (2007).

3.3.2 Order, LTEs and Consistency of BUM and BVM for $\mu = 4$

Since the BUM and the BVM are equally weighted, the orders and LTEs of the formulae constituting the BUM is equivalent for the BVM as well.

Expanding each of the formulae in (3.2.8), (3.2.10), (3.2.12) and (3.2.14), LTE, as defined in Definition 3.3.1 and in (3.3.8), is given as;

$$L[y(x); h] = C_{10} h^{10} y^{(10)}(x) + O(h^{11}); \text{ with } p = 6 \quad (3.3.9)$$

where $C_0 = C_1 = C_2 = \dots = C_{10} = 0$, and $C_9 \neq 0$, and $C_{10} = C_{p+4}$. Here p is the order and $\tau_{n+j} = C_{p+4}$, $j = 0(1)5$ is the error constant. The following table shows the order and error constants of each formulae in (3.2.8), (3.2.10), (3.2.12) and (3.2.14).

Table 3.1: Order and error constants for $\mu = 4$

Formulae	C_{p+4}	Formulae	C_{p+4}	Formulae	C_{p+4}
τ_{n+4}	$\frac{-1}{3024}$	-	-	-	-
τ_{n+5}	$\frac{-5}{3024}$	-	-	-	-
τ_n^I	$\frac{-1}{2880}$	τ_n^{II}	$\frac{167}{50400}$	τ_n^{III}	$\frac{-283}{24192}$
τ_{n+1}^I	$\frac{-163}{907200}$	τ_{n+1}^{II}	$\frac{630}{43200}$	τ_{n+1}^{III}	$\frac{311}{120960}$
τ_{n+2}^I	$\frac{-13}{50400}$	τ_{n+2}^{II}	$\frac{-31}{100800}$	τ_{n+2}^{III}	$\frac{151}{120960}$
τ_{n+3}^I	$\frac{-487}{907200}$	τ_{n+3}^{II}	$\frac{-29}{25200}$	τ_{n+3}^{III}	$\frac{-391}{120960}$
τ_{n+4}^I	$\frac{-307}{181440}$	τ_{n+4}^{II}	$\frac{251}{151200}$	τ_{n+4}^{III}	$\frac{89}{8064}$
τ_{n+5}^I		τ_{n+5}^{II}		τ_{n+5}^{III}	

where the formulae $\tau_{n+j}^{(m)}$, represent each LMM $y_{n+j}^{(m)}$, $m = 0, \dots, 3$; $j = 0, \dots, 5$.

From (3.3.9), it follows that for all the formulae, the order $p = 6$

3.3.3 Consistency of the Main Method

The first and second characteristic polynomials of the first formulae in (3.2.8) are:

$$\rho(z) = z^4 - 4z^3 + 6z^2 - 4z + 1$$

and

$$\sigma = \frac{1}{720} \cdot (-z^4 + 124z^3 + 474z^2 + 124z - 1)$$

By Definition 3.3.3 and Theorem 3.3.1, the first formulae in (3.2.8) is consistent since it satisfies the following:

1. The order is $p = 6 > 1$;

2. $\sum_{j=0}^4 a_j = 0$, ($a_0 = 1$, $a_1 = -4$, $a_2 = 6$, $a_3 = -4$, $a_4 = 1$)

3. $\rho(1) = 0$

$$\rho(1) = 4z^3 - 12z^2 + 12z - 4$$

$$\rho'(1) = 0$$

$$\text{Therefore, } \rho(1) = \rho'(1) = 0$$

$$4. \rho^{(iv)}(1) = 24; \sigma(1) = 1,$$

$$\text{Therefore, } \rho^{(iv)}(1) = 4!\sigma(1)$$

The first and second characteristic polynomials of the second formulae in (3.2.8) are;

$$\rho(z) = z^5 - 10z^3 + 20z^2 - 15z + 4$$

and

$$\sigma = \frac{1}{720} \cdot z^5 + 120z^4 - 970z^3 - 2020z^2 - 495z + 4$$

By Definition 3.3.3 and Theorem 3.3.1, the second formulae in (3.2.8) is consistent since it satisfies the following:

$$1. \text{ The order is } p = 6 > 1;$$

$$2. \sum_{j=0}^5 a_j = 0, (a_0 = 4, a_1 = -15, a_2 = 20, a_3 = -10, a_4 = 0, a_5 = 1)$$

$$3. \rho(1) = 0$$

$$\rho'(1) = 5z^4 - 30z^2 + 40z - 15$$

$$\rho''(1) = 0$$

$$\text{Therefore, } \rho(1) = \rho'(1) = 0$$

$$4. \rho^{(iv)}(1) = 120; \sigma(1) = 1,$$

$$\text{Therefore, } \rho^{(iv)}(1) = 5!\sigma(1)$$

By Definition 3.3.3, the block method which constitutes (3.2.8),(3.2.10),(3.2.12) and (3.2.14) is consistent.

3.3.4 Order, LTEs and Consistency of BUM and BVM for $\mu = 5$

Similarly, the order, LTE and Consistency of BUM for $\mu = 5$ suffice for that of the BVM for $\mu = 5$ as well, since both are equally weighted. Expanding each of the formulae in (3.2.22), (3.2.24), (3.2.26), (3.2.28) and (3.2.30), the LTE, as defined in Definition 3.3.1

and in (3.3.8), is given as;

$$L[y(x); h] = C_{10}h^{10}y^{(10)}(x) + O(h^{11}); \text{ with } p = 6 \quad (3.3.10)$$

where $C_0 = C_1 = C_2 = \dots = C_{10} = 0$, and $C_{11} \neq 0$, and $C_{11} = C_{p+5}$. Here p is the order and $\tau_{n+j} = C_{p+5}$, $j = 0(1)5$ is the error constant. The following table shows the order and error constants of formulae (3.2.22), (3.2.24), (3.2.26), (3.2.28) and (3.2.30).

Table 3.2: Order and error constants for $\mu = 5$

formulae	C_{p+5}	formulae	C_{p+5}	formulae	C_{p+5}	formulae	C_{p+5}
τ_{n+5}	$\frac{41}{221760}$	-	-	-	-	-	-
τ_n^j	$\frac{-79}{362880}$	τ_n^{jj}	$\frac{-2321}{907200}$	τ_n^{jjj}	$\frac{13}{1152}$	$\tau_n^{(iv)}$	$\frac{-1}{6048}$
τ_{n+1}^j	$\frac{-19}{266112}$	τ_{n+1}^{jj}	$\frac{37}{226800}$	τ_{n+1}^{jjj}	$\frac{227}{453600}$	$\tau_{n+1}^{(iv)}$	$\frac{-361}{120960}$
τ_{n+2}^j	$\frac{1}{71280}$	τ_{n+2}^{jj}	$\frac{-17}{201600}$	τ_{n+2}^{jjj}	$\frac{-43}{453600}$	$\tau_{n+2}^{(iv)}$	$\frac{181}{120960}$
τ_{n+3}^j	$\frac{37}{1330560}$	τ_{n+3}^{jj}	$\frac{89}{907200}$	τ_{n+3}^{jjj}	$\frac{-23}{129600}$	$\tau_{n+3}^{(iv)}$	$\frac{-67}{40320}$
τ_{n+4}^j	$\frac{-5}{44352}$	τ_{n+4}^{jj}	$\frac{-571}{1814400}$	τ_{n+4}^{jjj}	$\frac{229}{907200}$	$\tau_{n+4}^{(iv)}$	$\frac{341}{120960}$
τ_{n+5}^j	$\frac{-91}{570240}$	τ_{n+5}^{jj}	$\frac{-1}{3780}$	τ_{n+5}^{jjj}	$\frac{-337}{113400}$	$\tau_{n+5}^{(iv)}$	$\frac{-277}{24192}$

where the formulae $\tau_{n+j}^{(m)}$, represent each LMM $y_{n+j}^{(m)}$, $m = 0, \dots, 4$; $j = 0, \dots, 5$.

From (3.3.10), it follows that for all the formulae the order $p = 6$

3.3.5 Consistency of the Main Method

The first and second characteristic polynomials of formula (3.2.22) are;

$$\rho(z) = z^5 - 5z^4 + 10z^3 - 10z^2 + 5z - 1$$

and

$$\sigma = \frac{1}{24} z^4 + 11z^3 + z^2 + z + 11^\Sigma$$

By Definition 3.3.3 and Theorem 3.3.1, the formulae (3.2.22) is consistent since it satisfies the following:

1. order is $p = 6 > 1$;

$$\sum_{j=0}^5 a_j = 0, (a_0 = -1, a_1 = 5, a_2 = -10, a_3 = 10, a_4 = -5, a_5 = 1)$$

$$3. \rho(1) = 0$$

$$\rho'(z) = 5z^4 - 20z^3 + 30z^2 - 20z + 5$$

$$\rho'(1) = 0$$

$$\text{Therefore, } \rho(1) = \rho'(1) = 0$$

$$4. \rho^{(v)}(1) = 120; \alpha(1) = 1,$$

$$\text{Therefore, } \rho^{(v)}(1) = 5! \alpha(1)$$

By Definition 3.3.3, the block method which constitute (3.2.22), (3.2.24), (3.2.26), (3.2.28) and (3.2.30) is consistent.

3.3.6 Convergence Analysis

In this section, the convergence of the BUM and BVM are discussed. Since the BUMs and the BVMs for $\mu = 4$ and $\mu = 5$, respectively, are equally weighted their convergence are established in a similar way, with some changes in the coefficients of the matrices. Their LTEs are also the same. The following theorem is used to establish the convergence for the methods.

Theorem 3.3.2. (Jator and Manathunga, 2018) Let $\bar{Y}$ be an approximation of the solution vector Y for the system obtained on a partition $\pi_N = \{a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b\}$, and $E = Y - \bar{Y}$. Define $e_i = |y(x_i) - y_i|$, $h e_i^{(\mu)} = |h y^{(\mu)}(x_i) - h y_i^{(\mu)}|, \dots, h e_i = |h y(x_i) - h y_i|$ for $i = 1, \dots, N$, where the exact solution $y(x) \in C^n[a, b]$. Then the block method (3.3.5) is a sixth-order convergent method. That is $\|E\|_\infty = O(h^6)$.

3.3.6.1 Proof of Convergence of the BUM for $\mu = 4$

In order to show that the Block Unification Method (3.3.5) for $\mu = 4$ converges, the main methods and additional methods can be compactly written as

$$GY - h^4 HF(Y) + C + \tau(h) = 0 \quad (3.3.11)$$

where G is a $4N \times 4N$ matrix defined by

$$G = \begin{pmatrix} tt_{11} & tt_{12} & tt_{13} & tt_{14} \\ tt_{21} & tt_{22} & tt_{23} & tt_{24} \\ tt_{31} & tt_{32} & tt_{33} & tt_{34} \\ tt_{41} & tt_{42} & tt_{43} & tt_{44} \end{pmatrix}$$

and the entries of G are $N \times N$ matrices given as

$$G_{11} = \begin{pmatrix} -3 & 3/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -15 & 20 & -10 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 11/6 & -3 & 3/2 & -1/3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -2 & 5 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -15 & 20 & -10 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \ddots & \ddots & \ddots & & & \ddots & & & \\ \cdot & \cdot & \cdot & \ddots & & & \ddots & & \ddots & & & \vdots \\ \cdot & \cdot & \cdot & & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 11/6 & -3 & 3/2 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 5 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -15 & 20 & -10 & 0 & 1 \end{pmatrix}$$

1

¹ tt_{11} is the matrix of repeated coefficients of y_{n+i} for $i = 1(1)4$, in (3.2.8) and (3.2.10). tt_{21} is the matrix of repeated coefficients of hy_{n+i}^f for $i = 1(1)4$, in (3.2.10). tt_{31} is the matrix of repeated coefficients of $h^2 y_{n+i}^{rr}$ for $i = 1(1)4$, in (3.2.12) and tt_{41} is the matrix of repeated coefficients of $h^3 y_{n+i}^{rrr}$ for $i = 1(1)4$, in (3.2.14).

$$\begin{array}{c}
 \square \\
 \begin{array}{cccccccccccc}
 19 & 599 & 1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 105 & 10080 & 720 & 2240 & 3600 & & & & & & & \\
 3091 & 2831 & 143 & -1391 & 23 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 4320 & 30240 & 2520 & 60450 & 6048 & & & & & & & \\
 73967 & -1261 & 7439 & -5549 & 883 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 60450 & 6048 & 30240 & 60480 & 60480 & & & & & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 31 & 79 & 31 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 180 & 120 & 180 & 720 & 180 & & & & & & & \\
 11 & 101 & 97 & 6 & 720 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 16 & 56 & 72 & 1 & 720 & & & & & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 0 & 0 & 0 & 100800 & 105 & 10080 & 720 & 2240 & 3600 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 1411 & 2831 & 143 & -1391 & 23 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 20160 & 4320 & 30240 & 2520 & 60450 & 6048 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 19151 & 73967 & -1261 & 7439 & -5549 & 883 & 0 & \cdots & 0 \\
 & & & 60450 & 60450 & 6048 & 30240 & 60480 & 60480 & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 0 & 0 & 0 & -1 & 31 & 79 & 31 & -1 & 0 & 0 & \cdots & 0 \\
 & & & 720 & 180 & 120 & 180 & 720 & & & & \\
 0 & 0 & 0 & -1 & 11 & 101 & 97 & 6 & -1 & 0 & \cdots & 0 \\
 & & & 180 & 16 & 36 & 72 & 720 & & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 . & & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & . \\
 0 & 0 & 0 & 0 & \cdots & 0 & 937 & 19 & 599 & 1 & -3 & 1 \\
 & & & & & & 100800 & 105 & 10080 & 720 & 2240 & 3600 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 1411 & 3091 & 2831 & 143 & -1391 & 23 \\
 & & & & & & 20160 & 4320 & 30240 & 2520 & 60450 & 6048 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 19151 & 73967 & -1261 & 7439 & -5549 & 883 \\
 & & & & & & 60450 & 60450 & 6048 & 30240 & 60480 & 60480 \\
 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 31 & 79 & 31 & -1 & 0 \\
 & & & & & & 720 & 180 & 120 & 180 & 720 & \\
 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 11 & 101 & 97 & 6 & -1 \\
 & & & & & & 180 & 16 & 36 & 72 & 720 & \\
 & & & & & & & & & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 2809 & 43 & -229 & 1 & -11 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 60450 & 1008 & 30240 & 432 & 33600 & & & & & & & \\
 -311 & -353 & 1 & -7 & 53 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 10080 & 6048 & 135 & 2880 & 151200 & & & & & & & \\
 173 & 11 & 61 & 17 & -11 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 2880 & 60 & 10080 & 10080 & 33600 & & & & & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 169 & 12821 & 7447 & 509 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 540 & 10080 & 15120 & 60480 & 3600 & & & & & & & \\
 1663 & 6847 & 60863 & 2473 & 2041 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 2240 & 2160 & 30240 & 5040 & 302400 & & & & & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 0 & & & \sqrt{1} & 2809 & 43 & -229 & 1 & -11 & 0 & \cdots & 0 \\
 & & & 5400 & 60450 & 1008 & 30240 & 432 & 33600 & & & \\
 0 & 0 & 0 & 169 & -311 & -353 & 1 & -7 & 53 & 0 & \cdots & 0 \\
 & & & 302400 & 10080 & 6048 & 135 & 2880 & 151200 & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 0 & & & \sqrt{41} & 173 & 11 & 61 & 17 & -11 & 0 & \cdots & 0 \\
 & & & 50400 & 2880 & 60 & 10080 & 10080 & 33600 & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 0 & & & \sqrt{671} & 169 & 12821 & 7447 & 509 & -1 & 0 & \cdots & 0 \\
 & & & 302400 & 540 & 10080 & 15120 & 60480 & 3600 & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 0 & 0 & 0 & -31 & 1663 & 6847 & 60863 & 2473 & 2041 & 0 & \cdots & 0 \\
 & & & 5400 & 2240 & 2160 & 30240 & 5040 & 302400 & & & \square
 \end{array} \\
 \square \\
 \begin{array}{cccccccccccc}
 : & & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & : \\
 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2809 & 43 & -229 & 1 & -11 \\
 & & & & & & & 5400 & 60450 & 1008 & 30240 & 432 \\
 0 & 0 & 0 & 0 & \cdots & 0 & -311 & -353 & 1 & -7 & 53 & 33600 \\
 & & & & & & 10080 & 6048 & 135 & 2880 & 151200 & 151200 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 169 & 173 & 11 & 61 & 17 & -11 \\
 & & & & & & 302400 & 2880 & 60 & 10080 & 10080 & 33600 \\
 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{41} & 169 & 12821 & 7447 & 509 & -1 \\
 & & & & & & 50400 & 540 & 10080 & 15120 & 60480 & 3600 \\
 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{671} & 1663 & 6847 & 60863 & 2473 & 2041 \\
 & & & & & & 302400 & 2240 & 2160 & 30240 & 5040 & 302400
 \end{array}
 \end{array}$$

$$H_{31} = \begin{pmatrix}
\frac{-1601}{20160} & \frac{1}{7560} & \frac{-11}{4320} & \frac{11}{10080} & \frac{-11}{60480} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{-53}{15120} & \frac{-773}{10080} & \frac{-53}{15120} & \frac{11}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{10427}{60480} & \frac{9901}{15120} & \frac{107}{1120} & \frac{-37}{7560} & \frac{11}{60480} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{-73}{30240} & \frac{-1601}{20160} & \frac{1}{7560} & \frac{-11}{4320} & \frac{11}{10080} & \frac{-11}{60480} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{11}{60480} & \frac{-53}{15120} & \frac{-773}{10080} & \frac{-53}{15120} & \frac{11}{60480} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{-1}{720} & \frac{10427}{60480} & \frac{9901}{15120} & \frac{107}{1120} & \frac{-37}{7560} & \frac{11}{60480} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{-179}{60480} & \frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{-11}{30240} & \frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032} & 0 & \cdots & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{-73}{30240} & \frac{-1601}{20160} & \frac{1}{7560} & \frac{-11}{4320} & \frac{11}{10080} & \frac{-11}{60480} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{11}{60480} & \frac{-53}{15120} & \frac{-773}{10080} & \frac{-53}{15120} & \frac{11}{60480} & . \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{720} & \frac{10427}{60480} & \frac{9901}{15120} & \frac{107}{1120} & \frac{-37}{7560} & \frac{11}{60480} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{-179}{60480} & \frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{-11}{30240} & \frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032}
\end{pmatrix}$$

$$\begin{aligned}
 H_{41} = & \begin{array}{cccccccccccccccc}
 \square & \frac{-14033}{60480} & \frac{-10453}{30240} & \frac{2683}{30240} & \frac{-1717}{60480} & \frac{251}{60480} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 & \frac{12721}{60480} & \frac{11009}{30240} & \frac{-547}{6048} & \frac{1517}{60480} & \frac{-211}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 \square & \frac{1763}{12096} & \frac{27851}{30240} & \frac{14107}{30240} & \frac{-2389}{60480} & \frac{251}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 & \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & \frac{-883}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 \square & \frac{4783}{60480} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & 0 & 0 & \frac{799}{60480} & \frac{-14033}{60480} & \frac{-10453}{30240} & \frac{2683}{30240} & \frac{-1717}{60480} & \frac{251}{60480} & 0 & \dots & 0 & \square \\
 \square & 0 & 0 & 0 & \frac{-67}{12096} & \frac{12721}{60480} & \frac{11009}{30240} & \frac{-547}{6048} & \frac{1517}{60480} & \frac{-211}{60480} & 0 & \dots & 0 & \square \\
 \square & 0 & 0 & 0 & \frac{127}{60480} & \frac{1763}{12096} & \frac{27851}{30240} & \frac{14107}{30240} & \frac{-2389}{60480} & \frac{251}{60480} & 0 & \dots & 0 & \square \\
 \square & 0 & & & \frac{0}{12096} & \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & \frac{-883}{60480} & 0 & \dots & 0 & \square \\
 \square & 0 & & & \frac{799}{60480} & \frac{4783}{60480} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} & 0 & \dots & 0 & \square \\
 & \cdot & & & & \ddots & & \ddots & & \ddots & & \ddots & \cdot & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{799}{60480} & \frac{-14033}{60480} & \frac{-10453}{30240} & \frac{2683}{30240} & \frac{-1717}{60480} & \frac{251}{60480} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-67}{12096} & \frac{12721}{60480} & \frac{11009}{30240} & \frac{-547}{6048} & \frac{1517}{60480} & \frac{-211}{60480} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{127}{60480} & \frac{1763}{12096} & \frac{27851}{30240} & \frac{14107}{30240} & \frac{-2389}{60480} & \frac{251}{60480} & \square \\
 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-67}{12096} & \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & \frac{-883}{60480} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{799}{60480} & \frac{4783}{60480} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} & \square
 \end{array}
 \end{aligned}$$

$$H_{ij} = \mathbf{0}_{N \times N}, i = 1(1)4, j = 2(1)4, i < j.$$

$$Y = (y(x_1), \dots, y(x_N), hy^I(x_1), \dots, hy^I(x_N), h^2y^{II}(x_1), \dots, h^2y^{II}(x_N), h^3y^{III}(x_1), h^3y^{III}(x_N))^T$$

$$F(Y) = (f_1, \dots, f_N)^T$$

$$C = \begin{pmatrix} -hy_0^I, -\frac{11y_0}{6}, 0, \dots, 0, -h^2y_0^{II}, 2y_0, 0, \dots, 0, -h^3y_0^{III}, y_0, y_0, -y_0, \\ 6y_0, -3y_0, 6y_0, y_0, -y_0, y_0, -y_0, -y_0, -y_0, y_0, 0, \dots, 0 \end{pmatrix}^T$$

$$\tau(h) = (\tau_1, \dots, \tau_N, h\tau_1^I, \dots, h\tau_N^I, h^2\tau_1^{II}, \dots, h^2\tau_N^{II}, h^3\tau_1^{III}, \dots, h^3\tau_N^{III})^T$$

where $\tau(h)$ is the local truncation error. We also define E as

$$E = \bar{Y} - Y = (e_1, \dots, e_N, he_1^I, \dots, he_N^I, h^2e_1^{II}, \dots, h^2e_N^{II}, h^3e_1^{III}, \dots, h^3e_N^{III})^T$$

where

$$\bar{Y} = (y_1, \dots, y_N, hy_1^I, \dots, hy_N^I, h^2y_1^{II}, \dots, h^2y_N^{II}, h^3y_1^{III}, \dots, h^3y_N^{III})^T$$

3.3.6.2 Proof of Theorem 3.3.2

. The proof follows from Jator and Manathunga (2018). Let the exact form of the system (3.3.5) formed from (3.2.7), (3.2.9), (3.2.11) and (3.2.13) be as given in (3.3.11), while the approximate form is defined by the block

$$G\hat{Y} - h^4Hf(\hat{Y}) + C = 0 \quad (3.3.12)$$

subtracting (3.3.11) from (3.3.12), gives

$$G(\hat{Y} - Y) - h^4 H(f(\hat{Y}) - f(Y)) = \tau(h) \quad (3.3.13)$$

Using $e = \hat{Y} - Y$, we can write (3.3.13) as

$$G e - h^4 H(f(\hat{Y}) - f(Y)) = \tau(h) \quad (3.3.14)$$

Using Jacobian matrix, we can approximate

$$f(\hat{Y}) = f(Y) + J_f(Y)(\hat{Y} - Y) + o\|\hat{Y} - Y\| \quad (3.3.15)$$

where J_f as the approximate value $\frac{f(\bar{Y}) - f(Y)}{\bar{Y} - Y} = J_f(Y)$. Without lost of generality, the first derivative term can be used so that

$$f(\bar{Y}) - f(Y) = J_f(Y)(\bar{Y} - Y) = J_f(Y) e = J_f e \quad (3.3.16)$$

where

$$J_f = \begin{bmatrix} J_{11} & \cdots & J_{14} \\ \vdots & \ddots & \vdots \\ J_{41} & \cdots & J_{44} \end{bmatrix} \quad \text{where } J_{ij} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{bmatrix}$$

whose entries J_{ij} are $N \times N$ matrices written as $J_{11} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{bmatrix}$;

$$J_{1,j+1} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1}{\partial y_N^{(j)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N}{\partial y_N^{(j)}} \end{bmatrix} \quad \text{for } j = 1, 2, 3$$

$$J_{i,j+1} = h^i \begin{bmatrix} \frac{\partial f_1^{(i)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1^{(i)}}{\partial y_N^{(j)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N^{(i)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N^{(i)}}{\partial y_N^{(j)}} \end{bmatrix}; \quad \text{for } j = 0, \dots, 3. \text{ and for } i = 2, \dots, 4$$

Substituting (3.3.16) in (3.3.14), we get

$$G e - h^4 H J_f e = \tau(h) \quad (3.3.17)$$

$$(G - h^4 H(J_f))e = \tau(h)$$

Now, consider the matrix

$$v = G - h^4 H J_f \quad (3.3.18)$$

we claim that v is invertible for sufficiently small h . First, we claim that G is invertible. To see this, since $G_{ii} = I$, for $i = 3, 4$ and $G_{ij} = 0$ for $i = 1, 2, 3, 4; j = 3, 4; i \neq j$, we thus write P as

$$\begin{pmatrix} G_{11} & 0 & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ G_{31} & 0 & G_{33} & 0 \\ G_{41} & 0 & 0 & G_{44} \end{pmatrix} \quad (3.3.19)$$

Observe that G_{11} contains nonzero diagonal submatrices which are identical, so its determinant exists, hence its non singular. This is true for G_{21} , G_{31} and G_{41} . Note also that G_{ii} , $i = 2, 3, 4$ are identity matrices. Therefore G has nonzero for all diagonal elements. It is known that a matrix with nonzero main diagonal is invertible.

Thus, G is a matrix with nonzero diagonal, and so G^{-1} exists. Now, (3.3.18) can be written as

$$|v| = |G - h^4 H J_f| = |G| |I - C| \quad (3.3.20)$$

where $C = h^4 H U_f G^{-1}$, then $|\lambda I - C| = 0$ is a characteristic polynomial of C , so that

$$|\lambda I - C| = (\lambda - \lambda_1) \cdot \dots \cdot (\lambda - \lambda_{4N})$$

where λ_i are eigenvalues of the matrix C . When $\lambda = 1$, we have

$$|I - C| = (1 - \lambda_1) \cdot \dots \cdot (1 - \lambda_{4N})$$

for $|I - C| = 0$, then each $\lambda_i \neq 0$. If $\hat{\lambda}_i$ is an eigenvalue of C , so is $h^4 \lambda_i$, thus we need $h^4 \lambda_i \neq 1$. So we choose h such that $h^4 \notin \frac{1}{\lambda_i} \cdot \hat{\lambda}_i$ are nonzero eigenvector of $HJ_F G^{-1}$. For such h , $|I - C| = 0$, so that

$$|v| = |G| |I - H J_F| = 0 \quad (3.3.21)$$

Hence v is invertible. Then

$$\begin{aligned} v e &= \tau(h) \\ e &= v^{-1} \tau(h) \\ ||e|| &= ||v^{-1} \tau(h)|| \\ &\leq ||v^{-1}|| ||\tau(h)|| \\ &\leq O(h^{-4}) O(h^{10}) \\ &\leq O(h^6) \end{aligned}$$

This shows that the Block Unification Method for $\mu = 4$ is 6^{th} order convergent.

3.3.6.3 Proof of Convergence of the BVM for $\mu = 4$

The proof for the convergence of the Boundary Value Method (3.3.5) for $\mu = 4$ follows the same way as that of the block unification method. The difference is the change in the coefficients of the matrices G and H . In what follows, the matrices G and H are defined:

$$G = \begin{pmatrix} tt_{11} & tt_{12} & tt_{13} & tt_{14} \\ tt_{21} & tt_{22} & tt_{23} & tt_{24} \\ tt_{31} & tt_{32} & tt_{33} & tt_{34} \\ tt_{41} & tt_{42} & tt_{43} & tt_{44} \end{pmatrix}$$

where the elements of G are $N \times N$ matrices given as

$$\begin{aligned}
 G_{11} = & \begin{pmatrix} -3 & 3/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -15 & 20 & -10 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -15 & 20 & -10 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -15 & 20 & -10 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -15 & 20 & -10 & 0 & 1 \end{pmatrix} \\
 G_{21} = & \begin{pmatrix} 1/2 & -1 & 1/6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1/2 & -1/3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -3/2 & 3 & -11/6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -7 & 19/2 & -13/3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 13/3 & -31/2 & 19 & -47/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 11/6 & -7 & 19/2 & -13/3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 13/3 & -31/2 & 19 & -47/6 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 11/6 & -7 & 19/2 & -13/3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 13/3 & -31/2 & 19 & -47/6 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 11/6 & -7 & 19/2 & -13/3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 13/3 & -31/2 & 19 & -47/6 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
G_{31} = & \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -4 & 5 & -2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -7 & 8 & -3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -10 & 11 & -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -7 & 8 & -3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & -10 & 11 & -4 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & -7 & 8 & -3 & 0 & \cdots & 0 & 0 \\ & 0 & 3 & -10 & 11 & -4 & 0 & \cdots & 0 & \\ & & & & & & & & & \ddots \\ 0 & & & & & & & & & 0 \\ \vdots & 0 & 0 & 0 & \cdots & 0 & 2 & -7 & 8 & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 3 & -10 & 11 & -3 \\ 0 & -3 & 3 & -1 & 0 & 0 & 0 & \cdots & 0 & -4 \\ & & & & & & & & & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -3 & 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -3 & 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 \end{bmatrix} \\
G_{41} = & \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}
\end{aligned}$$

$G_{22} = G_{33} = G_{44} = I$, where I is an $N \times N$ identity matrices, $G_{ij} = \mathbf{0}_{N \times N}$, $i = 1(1)4, j = 2(1)4, i < j$.

Similarly, let H be a $4N \times 4N$ matrix defined by

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}$$

where the elements of b are $N \times N$ matrices given as

$$\begin{aligned}
 H_{11} = & \begin{bmatrix} \frac{19}{105} & \frac{599}{10080} & \frac{1}{720} & \frac{-3}{2240} & \frac{1}{3600} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{3091}{4320} & \frac{2831}{30240} & \frac{143}{2520} & \frac{-1391}{60450} & \frac{23}{6048} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{73967}{60450} & \frac{-1261}{6048} & \frac{7439}{30240} & \frac{-5549}{60480} & \frac{883}{60480} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{31}{180} & \frac{79}{120} & \frac{31}{180} & \frac{-1}{720} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{11}{16} & \frac{101}{36} & \frac{97}{72} & \frac{1}{6} & \frac{-1}{720} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{720} & \frac{31}{180} & \frac{79}{120} & \frac{31}{180} & \frac{-1}{720} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{180} & \frac{11}{16} & \frac{101}{36} & \frac{97}{72} & \frac{1}{6} & \frac{-1}{720} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & & \frac{1}{720} & \frac{31}{180} & \frac{79}{120} & \frac{31}{180} & \frac{-1}{720} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & & \frac{1}{180} & \frac{11}{16} & \frac{101}{36} & \frac{97}{72} & \frac{1}{6} & \frac{-1}{720} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{180} & \frac{31}{16} & \frac{79}{36} & \frac{31}{72} & \frac{-1}{720} & \frac{0}{720} \end{bmatrix} \\
H_{21} = & \begin{bmatrix} \frac{2809}{60450} & \frac{43}{1008} & \frac{-229}{30240} & \frac{1}{432} & \frac{-11}{33600} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-311}{10080} & \frac{-353}{6048} & \frac{1}{135} & \frac{-7}{2880} & \frac{53}{151200} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{173}{2880} & \frac{11}{60} & \frac{61}{10080} & \frac{17}{10080} & \frac{-11}{33600} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{169}{540} & \frac{12821}{10080} & \frac{7447}{15120} & \frac{509}{60480} & \frac{-1}{3600} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1663}{2240} & \frac{6847}{2160} & \frac{60863}{30240} & \frac{2473}{5040} & \frac{2041}{302400} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-671}{302400} & \frac{169}{540} & \frac{12821}{10080} & \frac{7447}{15120} & \frac{509}{60480} & \frac{-1}{3600} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-31}{5400} & \frac{1663}{2240} & \frac{6847}{2160} & \frac{60863}{30240} & \frac{2473}{5040} & \frac{2041}{302400} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-671}{302400} & \frac{169}{540} & \frac{12821}{10080} & \frac{7447}{15120} & \frac{509}{60480} & \frac{-1}{3600} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-31}{5400} & \frac{1663}{2240} & \frac{6847}{2160} & \frac{60863}{30240} & \frac{2473}{5040} & \frac{2041}{302400} & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-671}{302400} & \frac{169}{540} & \frac{12821}{10080} & \frac{7447}{15120} & \frac{509}{60480} & \frac{-1}{3600} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-31}{5400} & \frac{1663}{2240} & \frac{6847}{2160} & \frac{60863}{30240} & \frac{2473}{5040} & \frac{2041}{302400} \end{bmatrix}
\end{aligned}$$

$$H_{31} = \begin{pmatrix} \frac{-1601}{20160} & \frac{1}{7560} & \frac{-11}{4320} & \frac{11}{10080} & \frac{-11}{60480} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-53}{15120} & \frac{-773}{10080} & \frac{-53}{15120} & \frac{11}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{10427}{60480} & \frac{9901}{15120} & \frac{107}{1170} & \frac{-37}{7560} & \frac{11}{60480} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-179}{60480} & \frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-11}{30240} & \frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-179}{60480} & \frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-11}{30240} & \frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032} & 0 & 0 & \cdots & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-179}{60480} & \frac{3469}{10080} & \frac{6420}{4320} & \frac{3791}{3780} & \frac{121}{1344} & \frac{-23}{6048} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-11}{30240} & \frac{29689}{60480} & \frac{499}{210} & \frac{58271}{30240} & \frac{4561}{4320} & \frac{271}{4032} \end{pmatrix}$$

$$\begin{aligned}
 H_{41} = & \begin{array}{cccccccccccccccc}
 \frac{-14033}{60480} & \frac{-10453}{30240} & \frac{2683}{30240} & \frac{-1717}{60480} & \frac{251}{60480} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & & \\
 \frac{12721}{60480} & \frac{11009}{30240} & \frac{-547}{6048} & \frac{1517}{60480} & \frac{-211}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 \frac{1763}{12096} & \frac{27851}{30240} & \frac{14107}{30240} & \frac{-2389}{60480} & \frac{251}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & \frac{-883}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 \frac{4783}{6} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 & -12096 & \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & \frac{-883}{60480} & 0 & 0 & 0 & \dots & 0 & & \\
 & 60480 & \frac{4783}{60480} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} & 0 & 0 & 0 & \dots & 0 & & \\
 & 0 & \frac{-67}{12096} & \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & = & 0 & 0 & \dots & 0 & & \\
 & 0 & \frac{799}{60480} & \frac{4783}{60480} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} & 0 & 0 & \dots & 0 & & \\
 & : & & & & & & & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-67}{12096} & \frac{12049}{60480} & \frac{22433}{30240} & \frac{35569}{30240} & \frac{4873}{12096} & \frac{883}{60480} & \\
 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{799}{60480} & \frac{4783}{60480} & \frac{6511}{6048} & \frac{18811}{30240} & \frac{84299}{60480} & \frac{19067}{60480} &
 \end{array}
 \end{aligned}$$

$$H_{ij} = \mathbf{0}_{N \times N}, i = 1(1)4, j = 2(1)4, i < j.$$

Where the column matrix C is same as in the proof of the BUM.

3.3.6.4 Proof of Convergence of the BUM for $\mu = 5$

In order to show that the Block Unification Method (3.3.5) for $\mu = 5$ converges, the main methods and additional methods can be compactly written as

$$GY - h^5 HF(Y) + C + \tau(h) = 0 \quad (3.3.22)$$

where G is a $5N \times 5N$ matrix defined by

$$G = \begin{pmatrix} tt_{11} & tt_{12} & tt_{13} & tt_{14} & tt_{15} \\ tt_{21} & tt_{22} & tt_{23} & tt_{24} & tt_{25} \\ tt_{31} & tt_{32} & tt_{33} & tt_{34} & tt_{35} \\ tt_{41} & tt_{42} & tt_{43} & tt_{44} & tt_{45} \\ tt_{51} & tt_{52} & tt_{53} & tt_{54} & tt_{55} \end{pmatrix}$$

where G_{ij} are $N \times N$ matrices given as

$$G_{11} = \begin{bmatrix} -4 & 3 & -4/3 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 26/3 & -19/2 & 14/3 & -11/12 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -9 & 12 & 7 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 5 & -10 & 10 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25/12 & -4 & 3 & -4/3 & 1/4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -35/12 & 26/3 & -19/2 & 14/3 & -11/12 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/2 & -9 & 12 & 7 & 3/2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 5 & -10 & 10 & -5 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 25/12 & -4 & 3 & -4/3 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -35/12 & 26/3 & -19/2 & 14/3 & -11/12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 5/2 & -9 & 12 & 7 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -6 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

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$$G_{31} = \begin{pmatrix} \square & & & & & & & & & & & & \\ & 5/3 & -1/2 & -1/3 & 1/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \square \\ & -4/3 & 5/2 & -4/3 & 1/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \square \\ & -1/3 & -1/2 & -5/3 & -11/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \square \\ & 14/3 & -19/2 & 26/3 & -35/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \\ \square & 41/3 & -49/2 & 59/3 & 71/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \square \\ \square & 0 & 0 & 0 & -11/12 & 5/3 & -1/2 & -1/3 & 1/12 & 0 & \cdots & 0 & 0 & \square \\ & 0 & 0 & 0 & 1/12 & -4/3 & 5/2 & -4/3 & 1/12 & 0 & \cdots & 0 & 0 & \\ \square & 0 & 0 & 0 & 1/12 & -1/3 & -1/2 & -5/3 & -11/12 & 0 & \cdots & 0 & 0 & \square \\ \square & 0 & 0 & 0 & -11/12 & 14/3 & -19/2 & 26/3 & -35/12 & 0 & \cdots & 0 & 0 & \\ & 0 & 0 & 0 & -35/12 & 41/3 & -49/2 & 59/3 & 71/12 & 0 & \cdots & 0 & 0 & \square \\ \square & . & . & . & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & . & \square \\ \square & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -11/12 & 5/3 & -1/2 & -1/3 & 1/12 & \square \\ \square & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1/12 & -4/3 & 5/2 & -4/3 & 1/12 & \square \\ \square & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1/12 & -1/3 & -1/2 & -5/3 & -11/12 & \square \\ \square & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 11/12 & 14/3 & -19/2 & 26/3 & -35/12 & \square \\ \square & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -35/12 & 41/3 & -49/2 & 59/3 & 71/12 & \square \end{pmatrix}$$

Similarly, let H be a $5N \times 5N$ matrix defined by

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} \\ H_{21} & H_{22} & H_{23} & H_{24} & H_{25} \\ H_{31} & H_{32} & H_{33} & H_{34} & H_{35} \\ H_{41} & H_{42} & H_{43} & H_{44} & H_{45} \\ H_{51} & H_{52} & H_{53} & H_{54} & H_{55} \end{bmatrix}$$

where H_{ij} are $N \times N$ matrices given as

	$\square$	$\frac{107}{1440}$	$\frac{199}{1680}$	$\frac{3}{560}$	$\frac{19}{10080}$	$\frac{-1}{3360}$	0	0	0	0	0	0	0	...	0	$\square$	
	$\square$	$\frac{-21811}{60480}$	$\frac{-13079}{30240}$	$\frac{-41}{1440}$	$\frac{-209}{60480}$	$\frac{23}{43200}$	0	0	0	0	0	0	0	0	...	0	$\square$
	$\square$	$\frac{4081}{4320}$	$\frac{19433}{30240}$	$\frac{137}{1260}$	$\frac{-1061}{60480}$	$\frac{17}{6048}$	0	0	0	0	0	0	0	0	...	0	$\square$
	$\square$	$\frac{-81629}{60480}$	$\frac{-3799}{30240}$	$\frac{-8591}{30240}$	$\frac{-1061}{60480}$	$\frac{-853}{60480}$	0	0	0	0	0	0	0	0	...	0	$\square$
	$\square$	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0	0	0	0	0	0	0	0	0	...	0	$\square$
$H_{11} =$		0	0	0	0	$\frac{3360}{100800}$	$\frac{107}{60480}$	$\frac{199}{30240}$	$\frac{3}{1440}$	$\frac{19}{10080}$	$\frac{-1}{3360}$	0	0	0	...	0	
	$\square$	0	0	0	0	$\frac{(-) 887}{100800}$	$\frac{-21811}{60480}$	$\frac{-13079}{30240}$	$\frac{-41}{1440}$	$\frac{-209}{60480}$	$\frac{23}{43200}$	0	0	0	...	0	$\square$
		0	0	0	0	$\frac{277}{4032}$	$\frac{4081}{4320}$	$\frac{19433}{30240}$	$\frac{137}{1260}$	$\frac{-1061}{60480}$	$\frac{17}{6048}$	0	0	0	...	0	
	$\square$	0	0	0	0	$\frac{-19097}{60480}$	$\frac{-81629}{60480}$	$\frac{-3799}{30240}$	$\frac{-8591}{30240}$	$\frac{-1061}{60480}$	$\frac{-853}{60480}$	0	0	0	...	0	$\square$
		0	0	0	0	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0	0	0	0	...	0	
	$\square$	:	:	:	...	...	...	...	...	...	...	...	...	...	...	.	$\square$
		0	0	0	0	0	0	...	0	$\frac{3360}{10080}$	$\frac{107}{1440}$	$\frac{199}{1680}$	$\frac{3}{560}$	$\frac{19}{10080}$	=	$\frac{3360}{10080}$	
	$\square$	0	0	0	0	0	0	...	0	$\frac{1}{3360}$	$\frac{107}{1440}$	$\frac{199}{1680}$	$\frac{3}{560}$	$\frac{19}{10080}$	$\frac{-1}{3360}$	$\frac{-1061}{60480}$	$\frac{17}{6048}$
		0	0	0	0	0	0	...	0	$\frac{277}{4032}$	$\frac{4081}{4320}$	$\frac{19433}{30240}$	$\frac{137}{1260}$	$\frac{-1061}{60480}$	$\frac{-19097}{60480}$	$\frac{-81629}{60480}$	$\frac{-3799}{30240}$
	$\square$	0	0	0	0	0	0	...	0	$\frac{-19097}{60480}$	$\frac{-81629}{60480}$	$\frac{-3799}{30240}$	$\frac{-8591}{30240}$	$\frac{-1061}{60480}$	$\frac{-853}{60480}$	0	$\square$
		0	0	0	0	0	0	...	0	0						0	

⁴ H_{11} is the matrix of repeated coefficients of f_{n+i} , for $i = 0(1)5$, in (3.2.22). H_{21} is the matrix of repeated coefficients of f_{n+i} , $i = 0(1)5$, for all hy_{n+i}^I in (3.2.24). H_{31} is the matrix of repeated coefficients of f_{n+i} , $i = 0(1)5$, for all $h^2y_{n+i}^{II}$ in (3.2.26). H_{41} is the matrix of repeated coefficients of f_{n+i} , $i = 0(1)5$, for all $h^3y_{n+i}^{III}$ in (3.2.28). Finally, H_{51} is the matrix of repeated coefficients of f_{n+i} , $i = 0(1)5$, for all $h^4y_{n+i}^{(4)}$ in (3.2.30).

$$\begin{aligned}
 H_{21} = & \begin{bmatrix}
 \frac{1}{84} & \frac{-361}{10080} & \frac{-1}{480} & \frac{-1}{1680} & \frac{1}{10080} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 \frac{11}{2520} & \frac{25}{10080} & \frac{11}{2520} & \frac{-1}{10080} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 \frac{1}{280} & \frac{-341}{10080} & \frac{-3}{224} & \frac{1}{1120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 \frac{11}{1120} & \frac{9}{80} & \frac{397}{5040} & \frac{-1}{672} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 0 & \frac{1}{3360} & \frac{-1}{10080} & \frac{84}{2520} & \frac{361}{10080} & \frac{489}{2520} & \frac{1681}{10080} & \frac{1}{10080} & 0 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 0 & 0 & \frac{-1}{280} & \frac{-341}{10080} & \frac{-3}{224} & \frac{1}{1120} & \frac{-1}{10080} & 0 & 0 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 0 & \frac{1}{10080} & \frac{11}{1120} & \frac{9}{80} & \frac{397}{5040} & \frac{-1}{672} & \frac{1}{3360} & 0 & 0 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 0 & \frac{-1}{3360} & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & \cdots & 0 \\
 \cdot & \cdot & \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdot \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{10080} & \frac{1}{3360} & \frac{1}{10080} & \frac{1}{3360} & \frac{1}{10080} & \frac{1}{3360} & \frac{1}{10080} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{3360}{10080} & \frac{-1}{280} & \frac{84}{2520} & \frac{361}{10080} & \frac{489}{2520} & \frac{1681}{10080} & \frac{1}{10080} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{11}{2520} & \frac{10080}{10080} & \frac{25}{10080} & \frac{11}{2520} & \frac{-1}{1120} & \frac{1}{10080} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{10080} & \frac{11}{1120} & \frac{9}{80} & \frac{397}{5040} & \frac{-1}{672} & \frac{1}{3360} & \frac{1}{10080} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{3360} & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & \frac{1}{10080}
 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 H_{31} = & \begin{array}{cccccccccccccccc}
 \square & \frac{73}{2240} & \frac{779}{15120} & \frac{-13}{7560} & \frac{1}{576} & \frac{-79}{302400} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 & \frac{-337}{60480} & \frac{-17}{10080} & \frac{7}{864} & \frac{-11}{8640} & \frac{17}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 \square & \frac{-19}{8640} & \frac{-5}{108} & \frac{-23}{630} & \frac{127}{60480} & \frac{-79}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 & \frac{7}{192} & \frac{12757}{30240} & \frac{11147}{30240} & \frac{113}{20160} & \frac{23}{43200} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 \square & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & & & & 0 & & & & & & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & 0 & 0 & 0 & \frac{23}{43200} & \frac{2249}{60480} & \frac{15129}{10080} & \frac{-13}{7560} & \frac{-11}{8640} & \frac{-79}{302400} & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & & & & \frac{79}{302400} & \frac{-337}{60480} & \frac{-17}{10080} & \frac{7}{864} & \frac{-11}{8640} & \frac{17}{100800} & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & & & & \frac{17}{100800} & \frac{-19}{8640} & \frac{-5}{108} & \frac{-23}{630} & \frac{127}{60480} & \frac{-79}{100800} & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & 0 & 0 & 0 & \frac{79}{302400} & \frac{7}{192} & \frac{12757}{30240} & \frac{11147}{30240} & \frac{113}{20160} & \frac{23}{43200} & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & & & & \frac{23}{43200} & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & \cdot & \cdot & \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdot & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{43200}{43200} & \frac{2249}{60480} & \frac{15129}{10080} & \frac{-13}{7560} & \frac{-11}{8640} & \frac{-79}{302400} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{79}{302400} & \frac{-337}{60480} & \frac{-17}{10080} & \frac{7}{864} & \frac{-11}{8640} & \frac{17}{100800} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-17}{100800} & \frac{-19}{8640} & \frac{-5}{108} & \frac{-23}{630} & \frac{127}{60480} & \frac{-79}{100800} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{79}{302400} & \frac{7}{192} & \frac{12757}{30240} & \frac{11147}{30240} & \frac{113}{20160} & \frac{23}{43200} & \square \\
 \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-23}{43200} & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & \square
 \end{array}
 \end{aligned}$$

$H_{41} =$

$\square$	$\frac{31}{1344}$	$\frac{3251}{15120}$	$\frac{49}{4320}$	$\frac{41}{10080}$	$\frac{-41}{60480}$	0	0	0	0	0	0	0	0	...	0	$\square$
	$\frac{-419}{15120}$	$\frac{-395}{2016}$	$\frac{-419}{15120}$	$\frac{41}{60480}$	0	0	0	0	0	0	0	0	0	...	0	
$\square$	$\frac{1301}{60480}$	$\frac{1523}{7560}$	$\frac{67}{2016}$	$\frac{-13}{1890}$	$\frac{41}{60480}$	0	0	0	0	0	0	0	0	...	0	$\square$
	$\frac{671}{10080}$	$\frac{3019}{4320}$	$\frac{6823}{7560}$	$\frac{115}{1344}$	$\frac{-17}{60480}$	0	0	0	0	0	0	0	0	...	0	
$\square$	$\frac{5239}{60480}$	$\frac{6323}{5040}$	$\frac{54083}{30240}$	$\frac{4531}{4320}$	$\frac{277}{4032}$	0	0	0	0	0	0	0	0	...	0	$\square$
$\square$	0	0	0	0	$\frac{6045}{60480}$	$\frac{1344}{60480}$	$\frac{3251}{60480}$	$\frac{49}{60480}$	$\frac{41}{10080}$	$\frac{-41}{60480}$	0	0	0	...	0	$\square$
	0	0	0	0	$\frac{-419}{60480}$	$\frac{-395}{15120}$	$\frac{-419}{15120}$	$\frac{41}{60480}$	0	0	0	0	0	...	0	$\square$
$\square$	0	0	0	0	0	$\frac{1301}{60480}$	$\frac{1523}{7560}$	$\frac{67}{2016}$	$\frac{-13}{1890}$	$\frac{41}{60480}$	0	0	0	...	0	$\square$
$\square$	0	0	0	0	0	60480	7560	2016	1890	60480	0	0	0	...	0	$\square$
$\square$	0	0	0	0	$\frac{-41}{60480}$	$\frac{671}{10080}$	$\frac{3019}{4320}$	$\frac{6823}{7560}$	$\frac{115}{1344}$	$\frac{-17}{60480}$	0	0	0	...	0	$\square$
$\square$	0	0	0	0	$\frac{17}{6048}$	$\frac{5239}{60480}$	$\frac{6323}{5040}$	$\frac{54083}{30240}$	$\frac{4531}{4320}$	$\frac{277}{4032}$	0	0	0	...	0	$\square$
$\square$	0	0	0	0	...	...	...	...	...	...	...	...	...	...	0	$\square$
$\square$	0	0	0	0	0	0	0	...	0	$\frac{-17}{6045}$	$\frac{31}{1344}$	$\frac{3251}{15120}$	$\frac{49}{4320}$	$\frac{41}{10080}$	$\frac{-41}{60480}$	$\square$
$\square$	0	0	0	0	0	0	0	...	0	$\frac{41}{60480}$	$\frac{-419}{15120}$	$\frac{-395}{2016}$	$\frac{-419}{15120}$	$\frac{41}{60480}$	$\frac{-41}{60480}$	$\square$
$\square$	0	0	0	0	0	0	0	...	0	0	0	0	0	0	0	$\square$
$\square$	0	0	0	0	0	0	0	...	0	$\frac{671}{10080}$	$\frac{3019}{4320}$	$\frac{6823}{7560}$	$\frac{115}{1344}$	$\frac{-17}{60480}$	$\frac{17}{6048}$	$\square$
$\square$	0	0	0	0	0	0	0	...	0	$\frac{5239}{60480}$	$\frac{6323}{5040}$	$\frac{54083}{30240}$	$\frac{4531}{4320}$	$\frac{277}{4032}$	$\frac{277}{4032}$	$\square$

$$\begin{aligned}
 H_{51} = & \begin{bmatrix}
 \frac{-4339}{12096} & \frac{-20557}{30240} & \frac{1531}{30240} & \frac{-1867}{60480} & \frac{281}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 \frac{5059}{60480} & \frac{181}{6048} & \frac{-3887}{30240} & \frac{1367}{60480} & \frac{-181}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 \frac{1153}{60480} & \frac{17747}{30240} & \frac{2591}{6048} & \frac{-2539}{60480} & \frac{281}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 \frac{4387}{60480} & \frac{12329}{30240} & \frac{34417}{30240} & \frac{4843}{12096} & \frac{-853}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & 0 & 0 & \frac{853}{60480} & \frac{-4339}{12096} & \frac{-20557}{30240} & \frac{1531}{30240} & \frac{-1867}{60480} & \frac{281}{60480} & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & 0 & 0 & \frac{-281}{60480} & \frac{5059}{60480} & \frac{181}{6048} & \frac{-3887}{30240} & \frac{1367}{60480} & \frac{-181}{60480} & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & 0 & 0 & \frac{181}{60480} & \frac{1153}{60480} & \frac{17747}{30240} & \frac{2591}{6048} & \frac{-2539}{60480} & \frac{281}{60480} & 0 & 0 & 0 & \dots & 0 \\
 0 & & & & \frac{281}{60480} & \frac{4387}{60480} & \frac{12329}{30240} & \frac{34417}{30240} & \frac{4843}{12096} & \frac{-853}{60480} & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & 0 & 0 & \frac{853}{60480} & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & \dots & 0 \\
 \cdot & \cdot & \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{853}{60480} & \frac{-4339}{12096} & \frac{-20557}{30240} & \frac{1531}{30240} & \frac{-1867}{60480} & \frac{281}{60480} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-281}{60480} & \frac{5059}{60480} & \frac{181}{6048} & \frac{-3887}{30240} & \frac{1367}{60480} & \frac{-181}{60480} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{181}{60480} & \frac{1153}{60480} & \frac{17747}{30240} & \frac{2591}{6048} & \frac{-2539}{60480} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-281}{60480} & \frac{4387}{60480} & \frac{12329}{30240} & \frac{34417}{30240} & \frac{4843}{12096} & \frac{-853}{60480} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{853}{60480} & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480}
 \end{bmatrix}
 \end{aligned}$$

$$H_{ij} = \mathbf{0}_{N \times N}, i = 1(1)5, j = 2(1)5, i < j.$$

$$Y = (y(x_1), \dots, y(x_N), h y^{(1)}(x_1), \dots, h y^{(1)}(x_N), h^2 y^{(2)}(x_1), \dots, h^2 y^{(2)}(x_N), \\ h^3 y^{(3)}(x_1), h^3 y^{(3)}(x_N), h^4 y^{(iv)}(x_1), \dots, h^4 y^{(iv)}(x_N))^T$$

$$F(Y) = (f_1, \dots, f_N)^T$$

$$C = \begin{bmatrix} -h y_0, -\frac{25 y_0}{12}, h^5 \frac{f_0}{3360}, -h^2 y_0, \frac{35 y_0}{12}, h^5 \frac{-887 f_0}{100800}, -h^3 y_0, -\frac{5 y_0}{2}, h^5 \frac{277 f_0}{4032}, -h^4 y_0^{(4)}, y_0, h^5 \frac{-19097 f_0}{60480}, \\ y_0, 0, \dots, 0, -\frac{y_0}{4}, h^5 \frac{f_0}{3360}, \frac{y_0}{12}, -\frac{y_0}{12}, \frac{y_0}{4}, h^5 \frac{f_0}{10080}, \frac{25 y_0}{12}, h^5 \frac{-f_0}{3360}, 0, \dots, 0, \frac{11 y_0}{12}, h^5 \frac{79 f_0}{302400}, -\frac{y_0}{12}, \\ -h^5 \frac{17 f_0}{100800}, h^5 \frac{79 f_0}{302400}, \frac{35 y_0}{12}, h^5 \frac{-23 f_0}{43200}, 0, \dots, 0, -\frac{3 y_0}{2}, -h^5 \frac{17 f_0}{60480}, -\frac{y_0}{2}, h^5 \frac{41 f_0}{60480}, \frac{y_0}{2}, \frac{3 y_0}{2}, -h^5 \frac{41 f_0}{60480}, \\ \frac{5 y_0}{2}, h^5 \frac{17 f_0}{60480}, 0, \dots, 0, y_0, h^5 \frac{853 f_0}{60480}, y_0, h^5 \frac{-281 f_0}{60480}, y_0, h^5 \frac{181 f_0}{60480}, y_0, h^5 \frac{-281 f_0}{60480}, y_0, h^5 \frac{853 f_0}{60480}, 0, \dots, 0 \end{bmatrix} \Sigma_T$$

$$\tau(h) = (\tau_1, \dots, \tau_N, h \tau_1^{(1)}, \dots, h \tau_N^{(1)}, h^2 \tau_1^{(2)}, \dots, h^2 \tau_N^{(2)}, h^3 \tau_1^{(3)}, \dots, h^3 \tau_N^{(3)}, h^4 \tau_1^{(iv)}, \dots, h^4 \tau_N^{(iv)})^T$$

where $\tau(h)$ is the LTE. We also define E as

$$E = \bar{Y} - Y = (e_1, \dots, e_N, h e_1^{(1)}, \dots, h e_N^{(1)}, h^2 e_1^{(2)}, \dots, h^2 e_N^{(2)}, h^3 e_1^{(3)}, \dots, h^3 e_N^{(3)}, h^4 e_1^{(4)}, \dots, h^4 e_N^{(4)})^T$$

where

$$\bar{Y} = (y_1, \dots, y_N, h y_1^{(1)}, \dots, h y_N^{(1)}, h^2 y_1^{(2)}, \dots, h^2 y_N^{(2)}, h^3 y_1^{(3)}, \dots, h^3 y_N^{(3)}, h^4 y_1^{(iv)}, \dots, h^4 y_N^{(iv)})^T$$

3.3.6.5 Proof of Theorem 3.3.2

. The proof follows from Jator and Manathunga (2018), let the exact form of the system (3.3.5) formed from (3.2.21), (3.2.23), (3.2.25), (3.2.27) and (3.2.29) be as given in (3.3.22), while the approximate form be defined by the block

$$G\hat{Y} - h^5 H f(\hat{Y}) + C = 0 \quad (3.3.23)$$

Subtracting (3.3.22) from (3.3.23), gives

$$G(\hat{Y} - Y) - h^5 H(f(\hat{Y}) - f(Y)) = \tau(h) \quad (3.3.24)$$

Using the fact that $e = \hat{Y} - Y$, (3.3.24) can be written as

$$G e - h^5 H(f(\hat{Y}) - f(Y)) = \tau(h) \quad (3.3.25)$$

Using Jacobian matrix, we can approximate

$$f(\hat{Y}) = f(Y) + J_f(Y)(\hat{Y} - Y) + o\|\hat{Y} - Y\| \quad (3.3.26)$$

where J_f as the approximate value $\frac{f(\bar{Y}) - f(Y)}{\bar{Y} - Y} = J_f(Y)$. Without lost of generality, the first derivative term can be used so that

$$f(\bar{Y}) - f(Y) = J_f(Y)(\bar{Y} - Y) = J_f(Y) e = J_f e \quad (3.3.27)$$

where

$$J_f = \begin{bmatrix} J_{11} & \cdots & J_{15} \\ \vdots & \ddots & \vdots \\ J_{51} & \cdots & J_{55} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{bmatrix}$$

whose entries J_{ij} are $N \times N$ matrices written as $J_{11} = \begin{bmatrix} \cdot & \cdots & \cdot \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{bmatrix};$

$$J_{1,j+1} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1}{\partial y_N^{(j)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N}{\partial y_N^{(j)}} \end{bmatrix} \text{ for } j = 1, 2, 3, 4$$

$$J_{i,j+1} = h^i \begin{bmatrix} \frac{\partial f_1^{(j)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1^{(j)}}{\partial y_N^{(j)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N^{(j)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N^{(j)}}{\partial y_N^{(j)}} \end{bmatrix}; \text{ for } j = 0, \dots, 4. \text{ and for } i = 2, \dots, 5$$

Substituting (3.3.27) in (3.3.25) gives

$$G e - h^5 H J_f e = \tau(h) \quad (3.3.28)$$

$$(G - h^5 H(J_f))e = \tau(h)$$

Now, consider the matrix

$$v = G - h^5 H J_f \quad (3.3.29)$$

we claim that P is invertible for sufficiently small h . First, we claim that G is invertible.

To see this, since $G_{ii} = I$, for $i = 2(1)5$ and $G_{ij} = \mathbf{0}_{N \times N}$ for $i = 1(1)5; j = 2(1)5;$

$i \neq j$, we thus write G as

$$G = \begin{pmatrix} G_{11} & 0 & 0 & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 & 0 \\ G_{31} & 0 & G_{33} & 0 & 0 \\ G_{41} & 0 & 0 & G_{44} & 0 \\ G_{51} & 0 & 0 & 0 & G_{55} \end{pmatrix} \quad (3.3.30)$$

Observe that G_{11} contains non-zero diagonal matrices which are identical, so its determinant exists, hence it is non-singular. This is true for G_{21} , G_{31} , G_{41} and G_{51} . Note also that G_{ii} , $i = 2(1)5$ are identity matrices. Therefore G has all its diagonal elements non-zero.

Thus G is a matrix with nonzero diagonal, and so G^{-1} exists. Now, (3.3.29) can be written as

$$|v| = |G - h^5 H J_f| = |G| |I - C| \quad (3.3.31)$$

where $C = h^5 H J_f G^{-1}$, then $|\lambda I - C| = 0$ is a characteristic polynomial of C , so that

$$|\lambda I - C| = (\lambda - \lambda_1) \cdot \dots \cdot (\lambda - \lambda_{5N})$$

where λ_i are eigenvalues of the matrix C . When $\lambda = 1$, we have

$$|I - C| = (1 - \lambda_1) \cdots (1 - \lambda_{5N})$$

for $|I - C| = 0$, then each $\lambda_i \neq 0$. If $\hat{\lambda}_i$ is an eigenvalue of C , so is $h^5 \lambda_i$, thus we need $h^5 \lambda_i \neq 1$. So we choose h such that $h^5 \notin \frac{1}{\lambda_i} \cdot \hat{\lambda}_i$ are nonzero eigenvector of $HJ_F G^{-1}$. For such h , $|I - C| \neq 0$, so that

$$|v| = |G| |I - h^5 H J_F| \neq 0 \quad (3.3.32)$$

Hence v is invertible. Then

$$\begin{aligned} v e &= \tau(h) \\ e &= v^{-1} \tau(h) \\ \|e\| &= \|v^{-1} \tau(h)\| \\ &\leq \|v^{-1}\| \|\tau(h)\| \\ &\leq O(h^{-5}) O(h^{11}) \\ &\leq O(h^6) \end{aligned}$$

This shows that the Block Unification Method for $\mu = 5$ is 6th order convergent.

3.3.6.6 Proof of Convergence of the BVM for $\mu = 5$

The proof for the convergence of the Boundary Value Method (3.3.5) for $\mu = 5$ follows the same way as that of the block unification method. The difference is the change in the coefficients of the matrices G and H . In what follows, the matrices G and H are defined, where G is an $5N \times 5N$ matrix defined by

$$A = \begin{bmatrix} tt_{11} & tt_{12} & tt_{13} & tt_{14} & tt_{15} \\ tt_{21} & tt_{22} & tt_{23} & tt_{24} & tt_{25} \\ tt_{31} & tt_{32} & tt_{33} & tt_{34} & tt_{35} \\ tt_{41} & tt_{42} & tt_{43} & tt_{44} & tt_{45} \\ tt_{51} & tt_{52} & tt_{53} & tt_{54} & tt_{55} \end{bmatrix}$$

with G_{ij} an $N \times N$ matrices given as

$$\begin{array}{cccccccccccccccccccc}
& & -4 & 3 & -4/3 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
& & 26/3 & -19/2 & 14/3 & -11/12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
\begin{array}{l} \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} & & -9 & 12 & 7 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
& & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
& & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
& & 0 & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
& & 0 & 0 & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \\
& & 0 & 0 & 0 & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
& & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 5 & -10 & 10 & -5 & 0 & \\
& & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 5 & -10 & 10 & -5 &
\end{array}$$

$$\begin{array}{cccccccccccccccc}
& & 5/6 & 3/2 & 1/2 & -1/12 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & 2/3 & 0 & -2/3 & -1/12 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & -1/2 & 3/2 & -5/6 & -1/4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & 4/3 & -3 & 4 & 25/12 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & 0 & 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & \cdots & 0 & 0 & & & \\
& & 0 & 0 & 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 & \cdots & 0 & 0 & & & \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
& & 0 & 0 & 0 & 0 & \cdots & 0 & 61/6 & -39/2 & 107/6 & 77/12 & \cdots & & & \\
& & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 61/6 & -39/2 & 107/6 & 77/12 & & &
\end{array}$$

$$\begin{array}{cccccccccccccccccccc}
& & -\frac{1}{84} & -\frac{361}{10080} & -\frac{1}{480} & -\frac{1}{1680} & -\frac{1}{10080} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \square \\
& & \square & \frac{11}{2520} & \frac{25}{10080} & \frac{11}{2520} & \frac{1}{10080} & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \square \\
& & \square & \frac{280}{1120} & \frac{341}{10080} & \frac{224}{5040} & \frac{1120}{672} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \square \\
& & \square & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \square \\
& & \square & 0 & 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \square \\
& & \square & 0 & 0 & 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & \cdots & 0 & \square \\
& & \square & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \square \\
& & \square & & & & & & & & & & & & & & & 0 \\
& & \square & & & & & & & & & & & & & & & \frac{1680}{10080} \\
& & \square & \frac{149}{1680} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & - & 1680 & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} \\
& & \square & & & & & & & & & & & & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} \\
& & \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 10080 & 10080 & 180 & & \square
\end{array}$$

	$\frac{73}{2240}$	$\frac{779}{15120}$	$\frac{-13}{7560}$	$\frac{1}{576}$	$\frac{-79}{302400}$	0	0	0	0	0	0	0	...	0		
	$\frac{-337}{60480}$	$\frac{-17}{10080}$	$\frac{7}{864}$	$\frac{-11}{8640}$	$\frac{17}{100800}$	0	0	0	0	0	0	0	0	...	0	
	$\frac{-19}{8640}$	$\frac{-5}{108}$	$\frac{-23}{630}$	$\frac{127}{60480}$	$\frac{-79}{100800}$	0	0	0	0	0	0	0	0	...	0	
	$\frac{7}{192}$	$\frac{12757}{30240}$	$\frac{11147}{30240}$	$\frac{113}{20160}$	$\frac{23}{43200}$	0	0	0	0	0	0	0	0	...	0	
	$\frac{7559}{60480}$	$\frac{983}{720}$	$\frac{1672}{945}$	$\frac{29161}{60480}$	$\frac{887}{100800}$	0	0	0	0	0	0	0	0	...	0	
$H_{31} =$	0	$\frac{7559}{60480}$	$\frac{983}{7559}$	$\frac{1672}{983}$	$\frac{29161}{60480}$	$\frac{887}{100800}$	0	0	0	0	0	0	0	...	0	
	0	0	$\frac{7559}{60480}$	$\frac{983}{720}$	$\frac{1672}{945}$	$\frac{29161}{60480}$	$\frac{887}{100800}$	0	0	0	0	0	0	...	0	
	0	0	0	$\frac{7559}{60480}$	$\frac{983}{720}$	$\frac{1672}{945}$	$\frac{29161}{60480}$	$\frac{887}{100800}$	0	0	0	0	0	...	0	
	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	
										60480					0	
	0	0	0	0	0	0	0	...	0	$\frac{0}{7559}$	$\frac{7559}{60480}$	$\frac{983}{720}$	$\frac{1672}{945}$	$\frac{29161}{60480}$	$\frac{887}{100800}$	
	0	0	0	0	0	0	0	...	0							
											$\frac{983}{720}$	$\frac{1672}{945}$	$\frac{29161}{60480}$	$\frac{887}{100800}$		

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$$\begin{array}{cccccccccccccccccccc}
 & & \frac{-4339}{12096} & \frac{-20557}{30240} & \frac{1531}{30240} & \frac{-1867}{60480} & \frac{281}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & | \\
 \square & & \frac{5059}{60480} & \frac{181}{6048} & \frac{-3887}{30240} & \frac{1367}{60480} & \frac{-181}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 & & \frac{1153}{60480} & \frac{17747}{30240} & \frac{2591}{6048} & \frac{-2539}{60480} & \frac{281}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 \square & & \frac{4387}{60480} & \frac{12329}{30240} & \frac{34417}{30240} & \frac{4843}{12096} & \frac{-853}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 & & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \\
 H_{51} = \square & 0 & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & 0 & & & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \square \\
 \square & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \square \\
 & & & & & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & 60480 & & & & & & & \\
 \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-2879}{60480} & 60480 & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & & \\
 & & & & & & & & & & & & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & & \square \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-2879}{30240} & 30240 & 60480 & & & & \square
 \end{array}$$

where $H_{ij} = \mathbf{0}_{N \times N}$, $i = 1(1)5, j = 2(1)5, i < j$

$$C = \begin{bmatrix} -hy_0^r, -\frac{25y_0}{12}, h^5 \frac{f_0}{3360}, -h^2 y_0^r, \frac{35y_0}{12}, h^5 \frac{-887f_0}{100800}, -h^3 y_0^{rr}, -\frac{5y_0}{2}, h^5 \frac{277f_0}{4032}, -h^4 y_0^{(4)}, y_0, h^5 \frac{-19097f_0}{60480}, \\ y_0, 0, \dots, 0, -\frac{y_0}{4}, h^5 \frac{f_0}{3360}, \frac{y_0}{12}, -\frac{y_0}{12}, \frac{y_0}{4}, h^5 \frac{f_0}{100800}, \frac{25y_0}{12}, h^5 \frac{-f_0}{3360}, 0, \dots, 0, \frac{11y_0}{12}, h^5 \frac{-79f_0}{302400}, -\frac{y_0}{12} \\ -h^5 \frac{17f_0}{100800}, h^5 \frac{79f_0}{302400}, \frac{35y_0}{12}, h^5 \frac{-23f_0}{43200}, 0, \dots, 0, -\frac{3y_0}{2}, -h^5 \frac{17f_0}{6048}, -\frac{y_0}{2}, h^5 \frac{41f_0}{60480}, \frac{y_0}{2}, \frac{3y_0}{2}, -h^5 \frac{41f_0}{60480}, \\ \frac{5y_0}{2}, h^5 \frac{17f_0}{6048}, 0, \dots, 0, y, h^5 \frac{853f_0}{60480}, y, h^5 \frac{-281f_0}{60480}, y, h^5 \frac{181f_0}{60480}, y, h^5 \frac{-281f_0}{60480}, y, h^5 \frac{853f_0}{60480}, 0, \dots, 0 \end{bmatrix} \Sigma_T$$

CHAPTER FOUR: APPLICATION, RESULTS AND DISCUSSION

In this chapter, the application of all the methods derived using numerical examples shall be considered. The results obtained will be presented in tables and discussed appropriately.

4.1 Application of the Derived Methods

4.1.1 Fourth order Boundary Value Problem in Ordinary Differential Equation

In this section, the implementation of the BVM is illustrated with examples. The absolute errors are computed thus $Abs\ er = |y_{app} - y_{ex}|$. Maximum absolute errors are used for comparison where appropriate with other methods in literature, to show the efficiency of the method derived in this work, in terms of accuracy. In all the examples considered, the methods derived in this work, shows great deal of accuracy in comparison with other methods in the literature.

Problem 1. Consider the nonlinear two-point boundary value problem as (Costabile and Napoli, 2015):

$$\begin{aligned} y^{(iv)}(x) &= \sin(x) + \sin^2(x) - (y''(x))^2, \quad x \in [0, 1] \\ y(0) &= 0, \quad y'(0) = 1, \quad y(1) = \sin(1), \quad y'(1) = \cos(1) \end{aligned} \quad \square \quad (4.1.1)$$

with exact solution $y(x) = \sin(x)$.

Problem 2. Consider the linear boundary value problem (Costabile and Napoli, 2015):

$$\begin{aligned} y^{(iv)}(x) &= (1 + c)y''(x) - cy(x) + \frac{1}{2}cx^2 - 1, \quad x \in [0, 1] \\ y(0) &= 0, \quad y'(0) = 0, \quad y(1) = 1.5 + \sinh(1), \quad y'(1) = 1 + \cos(1) \end{aligned} \quad \square \quad (4.1.2)$$

with exact solution $y(x) = 1 + \frac{1}{2}x^2 + \sinh(x)$.

Problem 3. Consider the nonlinear boundary value problem (Xu, 2016):

$$\begin{aligned} y^{(iv)}(x) - y(x)y'(x) - 4x^7 - 24 &= 0, \quad x \in [0, 1] \\ y(0) &= 0, \quad y'''(0.25) = 6, \quad y''(0.5) = 3, \quad y(1) = 1 \end{aligned} \quad \square \quad (4.1.3)$$

with exact solution $y(x) = x^4$.

Problem 4. Consider the linear BVP (Costabile and Napoli, 2015):

$$\begin{aligned} y^{(iv)}(x) &= y(x) + y''(x) + e^x(x-3), x \in [0, 1] \\ y(0) &= 0, \quad y'(0) = 0, \quad y(1) = 3, \quad y'(1) = -e \end{aligned} \quad (4.1.4)$$

with exact solution $y(x) = (1-x)e^x$.

Problem 5. Consider the following nonlinear BVP (Xu, 2016):

$$\begin{aligned} y^{(iv)} + 6e^{-4y(x)} &= 0, \quad x \in [0, 4-e] \\ y(0) &= 1, \quad y(4-e) = \ln(4), \quad y''(0) = -\frac{1}{e^2}, \quad y''(4-e) = -\frac{1}{16} \end{aligned} \quad (4.1.5)$$

with exact solution $y(x) = \ln(x+e)$.

4.1.2 Fifth order Boundary Value Problem in Ordinary Differential Equation

In this section, the implementation of this method is demonstrated using examples in literature. The absolute errors are computed thus: $Abs\ er = |app\ sol - ex\ sol|$. Absolute errors are compared with those of other methods in literature. To show the efficiency of the method derived in this work, in terms of accuracy, maximum absolute errors are compared where applicable, with others in literature.

Problem 6. We consider the nonlinear fifth order boundary value problem (Ghazala and Hamood, 2013; Taiwo *et al.*, 2015):

$$\begin{aligned} y^{(v)}(x) + y^{(4)}(x) + e^{-2x}y^2 &= 2e^x + 1, \quad 0 \leq x \leq 1 \\ y(0) &= 1, \quad y(1) = e, \quad y'(0) = 1, \quad y'(1) = e, \quad y''(0) = 1 \end{aligned} \quad (4.1.6)$$

The analytic solution is given by $y(x) = e^x$.

Problem 7. Consider the fifth-order boundary value problem (Ghazala and Hamood, 2013; Gamel, 2007):

$$\begin{aligned} y^{(v)} + 24e^{-5y} &= \frac{48}{(1+x)^5}, \quad 0 < x < 1 \\ y(0) &= 0, \quad y(1) = \ln 2, \\ y'(0) &= 1, \quad y'(1) = 0.5, \\ y''(0) &= -1 \end{aligned} \quad (4.1.7)$$

Its analytic solution is $y(x) = \ln(1 + x)$

Problem 8. Consider the fifth-order boundary value problem, (Pandey, 2017).

$$\begin{aligned} & \square \quad y^{(5)}(x) - 4y'(x) = 0, \quad 0 < x < 1 \\ & \square \quad y(0) = 0, \quad y'(0) = 1, \\ & \quad y''(0) = 2, \quad y(1) = \exp(1) \sin(1), \\ & \square \quad y'(1) = \exp(1)(\sin(1) + \cos(1)) \end{aligned} \quad (4.1.8)$$

Its analytic solution is $y(x) = \exp(x) \sin(x)$.

Problem 9. Consider the fifth-order boundary value problem (Pandey, 2017):

$$\begin{aligned} & \square \quad y^{(5)}(x) - \frac{(y'(x))^2}{(5+x)^3} = \frac{23}{(5+x)^5}, \quad 0 < x < 1 \\ & \square \quad y(0) = \frac{1}{6} \\ & \square \quad y'(0) = -\frac{1}{25}, \quad y(1) = \ln(6) \sin(1), \\ & \square \quad y'(1) = \frac{6}{5} \end{aligned} \quad (4.1.9)$$

Its analytic solution is $y(x) = \ln(x + 5)$.

Problem 10. Consider the fifth-order boundary value problem (Doha *et al.*, 2015):

$$\begin{aligned} & \square \quad y^{(5)}(x) = y(x) - 15e^x - 10xe^x, \quad 0 < x < 1 \\ & \square \quad y(0) = 0, \quad y(1) = 1 \\ & \quad y'(0) = 1, \quad y'(1) = -e, \\ & \square \quad y''(0) = 0. \end{aligned} \quad (4.1.10)$$

Its analytic solution is $y(x) = x(1 - x)e^x$.

4.1.3 Fourth order Initial-Boundary Value Problem in Partial Differential Equation

In this section, some numerical examples are presented and error comparisons are made between the method derived above and those in existing literatures. The code used was based on Newton's method which uses the feature FindRoot or NSolve for linear problems in Mathematica. In all examples, a uniform stepsize was used and the maximum absolute errors were computed as $\max |(y_{ij} - y(x_i, t_j))|$, $0 \leq i \leq M$, $0 \leq j \leq N$. where y_{ij} is the numerical approximation of the exact solution $y(x_i, t_j)$ at the mesh point (x_i, t_j) .

Problem 11. Consider the linear fourth-order parabolic equation with constant coefficients (Rashidinia and Mohammadi, 2010):

$$\begin{aligned} y_{tt} + y_{xxxx} &= (\pi^4 - 1) \sin \pi x \cos t, \quad 0 \leq x \leq 1, \quad t \geq 0 \\ y(x, 0) &= \sin \pi x, \quad y_t(x, 0) = 0 \\ y(0, t) &= y(1, t) = y_{xx}(0, t) = y_{xx}(1, t) = 0 \end{aligned} \quad (4.1.11)$$

with the exact solution $y(x, t) = \sin \pi x \cos t$

Here, following Ngwane and Jator (2017), the problem on discretizing the time variable, becomes

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + \frac{d^4 y_m}{dx^4} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (4.1.12)$$

where $\Delta t = (L_4 - L_3)/M$, $t_m = L_3 + m\Delta t$, $m = 0, 1, \dots, M$, $y = [y_1(x), \dots, y_M(x)]^T$, $y_m(x) \approx y(x, t_m)$, $g = [g_1(x), \dots, g_M(x)]^T$ and $g_m(x) \approx g(x, t_m) = (\pi^4 - 1) \sin \pi x \cos t_m$, which is expressed in the form

$$y^{(iv)} = f(x, y, y', y'', y''') = Ay + g \quad (4.1.13)$$

A is an $(M-1) \times (M-1)$ matrix arising from the discretized system, and g is a vector of constants.

Problem 12. Consider the homogenous fourth-order parabolic equation (Rashidinia and Mohammadi, 2010):

$$\begin{aligned} 120xy_{tt} + (120 + x^5)y_{xxxx} &= 0, \quad \frac{1}{2} \leq x \leq 1, \quad t \geq 0 \\ y(x, 0) &= \frac{3841}{3840} \sin t, \quad y_t(x, 0) = \frac{x^5 121}{120} \sin t \\ y_{xx}(\frac{1}{2}, t) &= \frac{1}{48} \sin t, \quad y_{xx}(\frac{1}{2}, t) = \frac{1}{8} \sin t \end{aligned} \quad (4.1.14)$$

The exact solution for this problem is $y(x, t) = 1 + \frac{x^5}{120} \sin t$.

Upon discretization of the time variable, we have

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + \frac{(120 + x^5)}{120x} \frac{d^4 y_m}{dx^4} = g_m, \quad \frac{1}{2} \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (4.1.15)$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, which is put in the form

$$y^{(iv)} = f(x, y, y', y'', y''') = Ay + g \quad (4.1.16)$$

A is as expressed in Problem (4.1.11) and $g_m = 0$.

Problem 13. Consider the homogenous fourth-order parabolic equation (Rashidinia and Mohammadi, 2010):

$$\begin{aligned}
 (\sin x)y_{tt} + (x - \sin x)y_{xxxx} &= 0, \quad \frac{1}{2} \leq x \leq 1, \quad t \geq 0 \quad \square \\
 y(x, 0) &= x - \sin x, \quad y_t(x, 0) = -(x - \sin x) \quad \square \\
 y(0, t) &= 0, \quad y(1, t) = e^{-t}(1 - \sin 1) \\
 y_{xx}(0, t) &= 0, \quad y_{xx}(1, t) = e^{-t} \sin 1 \quad \square
 \end{aligned} \tag{4.1.17}$$

The exact solution for this problem is $y(x, t) = (x - \sin x)e^{-t}$.

Upon discretization of the time variable, we have

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + \frac{(x - \sin x)}{\sin x} \frac{d^4 y_m}{dx^4} = g_m, \quad \frac{1}{2} \leq x \leq 1, \quad m = 1, \dots, M-1 \tag{4.1.18}$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1 which is put in the form

$$y^{(iv)} = f(x, y, y^r, y^{rr}, y^{rrr}) = Ay + g \tag{4.1.19}$$

A is as expressed in Problem (4.1.11) and $g_m = 0$

Problem 14. Consider the nonhomogenous fourth-order parabolic equation (Rashidinia and Mohammadi, 2010):

$$\begin{aligned}
 y_{tt} + (1 + x)y_{xxxx} &= x^3 + x^4 - \frac{6}{7!}x^7 \cos t, \quad \frac{1}{2} \leq x \leq 1, \quad t \geq 0 \quad \square \\
 y(x, 0) &= \frac{6}{7!}x^7, \quad y_t(x, 0) = 0 \quad \square \\
 y(0, t) &= 0, \quad y(1, t) = \frac{6}{7!} \cos t \\
 y_{xx}(0, t) &= 0, \quad y_{xx}(1, t) = \frac{1}{20} \cos t \quad \square
 \end{aligned} \tag{4.1.20}$$

The exact solution for this problem is $y(x, t) = \frac{6}{7!}x^7 \cos t$.

Upon discretization of the time variable, we have

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + (1 + x) \frac{d^4 y_m}{dx^4} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \tag{4.1.21}$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, $g_m = x^3 + x^4 - \frac{6}{7!}x^7 \cos t_m$, which is expressed in the form

$$y^{(iv)} = f(x, y, y^r, y^{rr}, y^{rrr}) = Ay + g \tag{4.1.22}$$

A is as expressed in Problem (4.1.11).

Problem 15. Consider the "good" Boussinesq equation from Mohanty & Kaur (2016) and Siddiqi & Arshed (2014):

$$\begin{aligned} y_{tt} &= y_{xx} + y_{xx}^2 - y_{xxxx}, & 0 \leq x \leq 1, \quad t \geq 0 \\ y(0, t) &= 0, \quad y(1, t) = 0, \\ y_{xx}(0, t) &= 0, \quad y_{xx}(1, t) = 0, \quad t > 0 \end{aligned} \quad \square \quad (4.1.23)$$

The exact solution for this problem is $y(x, t) = -A \operatorname{sech} h^2 \left[\frac{A}{b} (x - ct + v_0) - b + 1 \right]^{\frac{1}{2}}$.

Here c is the velocity, A is amplitude of the pulse, b is an arbitrary parameter and v_0 is the initial position. Using the same theoretical parameters as in Mohanty and Kaur (2016): $A = 0.369$, $b = -1/2$ and $c = 0.868$.

Here, the problem on discretizing the time variable becomes

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} - \frac{d^4 y_m}{dx^4} - \frac{\partial^2 y_m^2}{\partial x^2} + \frac{\partial^4 y_m}{\partial x^4} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (4.1.24)$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, which is expressed in the form

$$y^{(iv)} = f(x, y, y^r, y^{rr}, y^{rrr}) = Ay + g \quad (4.1.25)$$

A is as expressed in problem (4.1.11) and $g_m = 0$.

4.1.4 Fifth order Initial-Boundary Value Problem in Partial Differential Equation

Problem 16. Consider a time dependent biharmonic fifth order equation (Cheng and Shu, 2008):

$$\begin{aligned} y_t + y_x + y_{xxxxx} &= 0, & 0 \leq x \leq 2\pi, \quad t \geq 0 \\ y(x, 0) &= \sin(x), \\ y(0, t) &= y(2\pi, t) \end{aligned} \quad \square \quad (4.1.26)$$

The analytic solution for this problem is $y(x, t) = \sin(x - 2t)$.

Upon discretization of the time variable, we obtain,

$$\frac{y_{m+1} - y_{m-1}}{(2\Delta t)} + \frac{dy_m}{dx} + \frac{d^5 y_m}{dx^5} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (4.1.27)$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y^r, y^{rr}, y^{rrr}, y^{(iv)}) = Ay + g \quad (4.1.28)$$

A is a $k \times k$ matrix ($k = (N-1)(M-1)$) and $g_m = 0$.

Problem 17. Consider the nonlinear fifth order KdV equation (Goswami *et al.*, 2017):

$$\begin{aligned} y_t + yy_x - yy_{xxx} + y_{xxxxx} &= 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \\ y(x, 0) &= e^x, \quad \square \end{aligned} \quad (4.1.29)$$

The exact solution for this problem is $y(x, t) = e^{x-t}$.

Upon discretization of the time variable, we obtain,

$$\frac{y_{m+1} - y_{m-1}}{(2\Delta t)} + y_m \frac{dy_m}{dx} - y_m \frac{d^3 y_m}{dx^3} + \frac{d^5 y_m}{dx^5} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (4.1.30)$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y^r, y^{rr}, y^{rrr}, y^{(iv)}) = Ay + g \quad (4.1.31)$$

A is as expressed in Problem (4.1.26) and $g_m = 0$.

Problem 18. Consider the nonlinear fifth order KdV equation called Lax's cases of generalized KdV equations (Mohyud-Din *et al.*, 2012):

$$\begin{aligned} y_t + 30y^2 y_x + y_x y_{xx} + 10y y_{xxx} + y_{xxxxx} &= 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \\ y(x, 0) &= 2k^2(2 - 3 \tanh(k(x - x_0))) \quad \square \end{aligned} \quad (4.1.32)$$

The exact solution for this problem is $y(x, t) = 2k^2(2 - 3 \tanh(k(x - 56k^4t - x_0)))$

Upon discretization of the time variable, we obtain,

$$\begin{aligned} \frac{y_{m+1} - y_{m-1}}{(2\Delta t)} + 30y_m^2 \frac{dy_m}{dx} + y_m \frac{d^2 y_m}{dx^2} + 10y_m \frac{d^3 y_m}{dx^3} + \frac{d^5 y_m}{dx^5} &= g_m \\ 0 \leq x \leq 1, \quad m &= 1, \dots, M-1 \end{aligned} \quad (4.1.33)$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y^r, y^{rr}, y^{rrr}, y^{(iv)}) = Ay + g \quad (4.1.34)$$

A is as expressed in Problem (4.1.26) and $g_m = 0$.

Problem 19. Consider the nonlinear fifth order KdV equation called Lax's cases of generalized KdV equations (Mohyud-Din *et al.*, 2012):

$$\begin{aligned} y_t + 45y^2 y_x + 15y_x y_{xx} + 15y y_{xxx} + y_{xxxxx} &= 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \\ y(x, 0) &= 2k^2 \sec h^2(k(x - x_0)) \quad \square \end{aligned} \quad (4.1.35)$$

The exact solution for this problem is $y(x, t) = 2k^2 \sec h^2(k(x - 16k^4t - x_0))$

Upon discretization of the time variable, we obtain,

$$\frac{y_{m+1}-y_{m-1}}{(2\Delta t)} + 45y_m y_m \frac{dy_m}{dx} + 15 \frac{dy_m}{dx} \frac{d^2 y_m}{dx^2} + 15y_m \frac{d^3 y_m}{dx^3} + \frac{d^5 y_m}{dx^5} = g, \quad m$$

$$0 \leq x \leq 1, \quad m = 1, \dots, M-1$$
(4.1.36)

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y^r, y^{rr}, y^{rrr}, y^{(iv)}) = Ay + g \quad (4.1.37)$$

A is as expressed in Problem (4.1.26) and $g_m = 0$.

Problem 20. Consider the nonlinear fifth order KdV equation called Caudrey-Dodd-Gibbon (CDG) cases of generalized KdV equations (Modebei *et al.*, 2015).

$$y_t + 180y^2 y_x + 30y_x y_{xx} + 30y y_{xxx} + y_{xxxxx} = 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \quad (4.1.38)$$

$$y(x, 0) = -\frac{1200}{\sec h^2 \frac{x}{20}} \quad \square$$

The exact solution for this problem is $y(x, t) = -\frac{1}{1200} \left(1 - 3 \sec h^2 \frac{x}{20} - \frac{0.0001t}{4} \right)$

Upon discretization of the time variable, we obtain,

$$\frac{y_{m+1}-y_{m-1}}{(2\Delta t)} + 180y_m y_m \frac{dy_m}{dx} + 30 \frac{dy_m}{dx} \frac{d^2 y_m}{dx^2} + 30y_m \frac{d^3 y_m}{dx^3} + \frac{d^5 y_m}{dx^5} = g, \quad m$$

$$0 \leq x \leq 1, \quad m = 1, \dots, M-1$$
(4.1.39)

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y^r, y^{rr}, y^{rrr}, y^{(iv)}) = Ay + g \quad (4.1.40)$$

A is as expressed in Problem (4.1.26) and $g_m = 0$.

4.1.5 Tables of Results

4.1.5.1 Table of results for fourth order BVPs

Table 4.1: Numerical Results for Problem 1

x	Exact	BUM	BVM
0.0	0.0	0.0	0.0
0.1	0.099833416646828	0.099833416646856	0.0998334166468286
0.2	0.198669330795061	0.198669330795143	0.1986693307950645
0.3	0.295520206661339	0.295520206661478	0.2955202066613464
0.4	0.389418342308650	0.389418342308834	0.3894183423086605
0.5	0.479425538604203	0.479425538604405	0.4794255386042147
0.6	0.564642473395035	0.564642473395226	0.5646424733950469
0.7	0.644217687237691	0.644217687237838	0.6442176872377005
0.8	0.717356090899522	0.717356090899609	0.7173560908995287
0.9	0.783326909627483	0.783326909627508	0.7833269096274855
1.0	0.841470984807896	0.841470984807896	0.8414709848078965

Table 4.2: Error of Methods for Problem 1

x	Error BUM	Error BVM	Er^{NMD} in Ref A	$Er^{HBV P}$ in Ref B
0.0	0.0	0.0	0.0	0.0
0.1	2.85×10^{-14}	5.36×10^{-16}	7.78×10^{-8}	4.45×10^{-10}
0.2	8.18×10^{-14}	3.31×10^{-15}	2.72×10^{-7}	5.54×10^{-10}
0.3	1.38×10^{-13}	6.88×10^{-15}	5.24×10^{-7}	8.95×10^{-11}
0.4	1.83×10^{-13}	1.00×10^{-14}	7.77×10^{-7}	2.03×10^{-10}
0.5	2.02×10^{-13}	1.17×10^{-14}	9.71×10^{-7}	3.32×10^{-11}
0.6	1.90×10^{-13}	1.15×10^{-14}	1.05×10^{-6}	1.53×10^{-10}
0.7	1.47×10^{-13}	9.48×10^{-15}	9.63×10^{-7}	9.48×10^{-11}
0.8	8.67×10^{-14}	5.97×10^{-15}	6.84×10^{-7}	5.18×10^{-10}
0.9	2.54×10^{-14}	2.20×10^{-15}	2.71×10^{-7}	4.15×10^{-10}
1.0	0.0	0.0	0.0	0.0

Ref A is Noor and Mohyud-Din (2007); Ref B is Costabile and Napoli (2015).

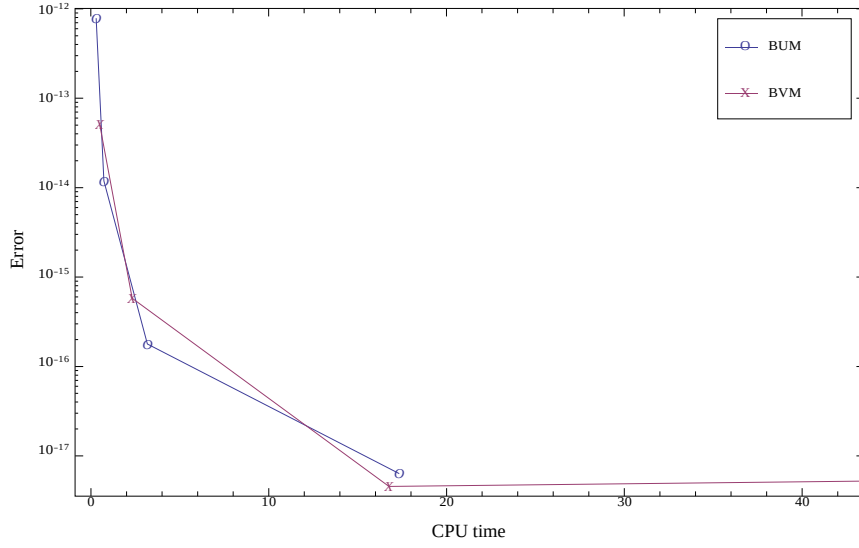


Figure 4.1.1: An efficiency graph of CPU time versus Errors for Problem 1

Table 4.3: Numerical Results for Problem 2

x	Exact	BUM	BVM
0.0	1.0	1.0	1.0
0.1	1.105166750019843	1.105166750019825	1.105166750019843
0.2	1.221336002541094	1.221336002541019	1.221336002541092
0.3	1.349520293447142	1.349520293447001	1.349520293447140
0.4	1.490752325802815	1.490752325802615	1.490752325802812
0.5	1.646095305493747	1.646095305493516	1.646095305493745
0.6	1.816653582148241	1.816653582148013	1.816653582148239
0.7	2.003583701839533	2.003583701839346	2.003583701839532
0.8	2.208105982187623	2.208105982187505	2.208105982187623
0.9	2.431516725708175	2.431516725708131	2.431516725708175
1.0	2.675201193643801	2.675201193643801	2.675201193643801

Table 4.4: Error of Methods for Problem 2

x	$Er1^{BUM}$	$Er1^{BV\ M}$	$Er1^{HBV\ P}$	$Er2^{BUM}$	$Er2^{BV\ M}$	$Er2^{HBV\ P}$	$Er3^{BUM}$	$Er3^{BV\ M}$	$Er3^{HBV\ P}$
0.1	2.36×10^{-14}	1.17×10^{-14}	5.44×10^{-10}	1.23×10^{-14}	2.11×10^{-14}	1.84×10^{-10}	1.36×10^{-14}	1.46×10^{-14}	3.18×10^{-9}
0.2	9.08×10^{-14}	3.77×10^{-14}	6.71×10^{-10}	5.11×10^{-14}	6.79×10^{-14}	6.43×10^{-10}	5.51×10^{-14}	4.88×10^{-14}	1.28×10^{-8}
0.3	1.73×10^{-13}	6.92×10^{-14}	9.72×10^{-11}	9.96×10^{-14}	1.20×10^{-13}	2.39×10^{-9}	1.04×10^{-13}	9.04×10^{-14}	2.54×10^{-8}
0.4	2.47×10^{-13}	9.45×10^{-14}	2.49×10^{-10}	1.44×10^{-13}	1.61×10^{-13}	3.67×10^{-9}	1.47×10^{-13}	1.26×10^{-13}	3.55×10^{-8}
0.5	2.87×10^{-13}	1.06×10^{-13}	1.85×10^{-11}	1.71×10^{-13}	1.80×10^{-13}	3.78×10^{-9}	1.71×10^{-13}	1.46×10^{-13}	3.89×10^{-8}
0.6	2.87×10^{-13}	9.81×10^{-14}	2.03×10^{-10}	1.76×10^{-13}	1.73×10^{-13}	3.18×10^{-9}	1.71×10^{-13}	1.43×10^{-13}	3.49×10^{-8}
0.7	2.35×10^{-13}	7.42×10^{-14}	1.65×10^{-10}	1.49×10^{-13}	1.39×10^{-13}	2.60×10^{-9}	1.43×10^{-13}	1.15×10^{-13}	2.56×10^{-8}
0.8	1.49×10^{-13}	4.21×10^{-14}	7.51×10^{-10}	9.87×10^{-14}	8.29×10^{-14}	2.04×10^{-9}	9.60×10^{-14}	7.05×10^{-14}	1.42×10^{-8}
0.9	5.31×10^{-13}	1.35×10^{-14}	5.90×10^{-10}	3.80×10^{-14}	2.51×10^{-14}	9.49×10^{-10}	3.49×10^{-14}	2.38×10^{-14}	4.31×10^{-9}

HBVP is the method in Costabile and Napoli (2015).

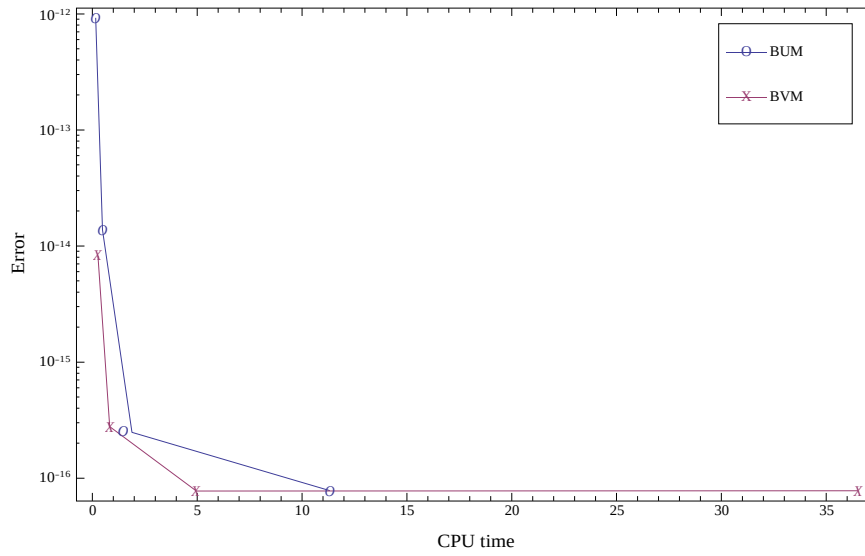


Figure 4.1.2: An efficiency graph of CPU time versus Errors for Problem 2

Table 4.5: Numerical Results for Problem 3

x	Exact	BUM	BVM
0.0	0	0	0
0.1	0.0000100000000025	0.0000999999999999	0.0000999999999999
0.2	0.0016000000000025	0.0015999999999999	0.0015999999999999
0.3	0.0081000000000025	0.0080999999999999	0.0080999999999999
0.4	0.0256000000000025	0.0255999999999999	0.0255999999999999
0.5	0.0625000000000025	0.0624999999999999	0.0624999999999999
0.6	0.1296000000000025	0.1295999999999999	0.1295999999999999
0.7	0.2401000000000025	0.2400999999999999	0.2400999999999999
0.8	0.4096000000000025	0.4095999999999999	0.4095999999999999
0.9	0.6561000000000025	0.6560999999999999	0.6560999999999999
1.0	1.0000000000000025	1.0000000000000000	1.0000000000000000

Table 4.6: Error of Methods for Problem 3

Tatari and Dehghan (2006)	Duan and Rach (2011)	Xu, <i>et al</i> (2016)	BUM	BVM
method 1: 2.2×10^{-9}	$k_1 = 1$: 3.4×10^{-4}	method 1: 1.8×10^{-12}	N=5: 4.99×10^{-41}	N=5: 4.49×10^{-41}
method 2: 1.3×10^{-9}	$k_1 = 2$: 7.1×10^{-7}	method 2: 5.6×10^{-13}	N=10: 1.56×10^{-40}	N=10: 1.47×10^{-40}
method 3: 1.0×10^{-10}	$k_1 = 3$: 5.4×10^{-9}	method 3: 2.2×10^{-14}	N=15: 3.99×10^{-39}	N=15: 9.07×10^{-39}

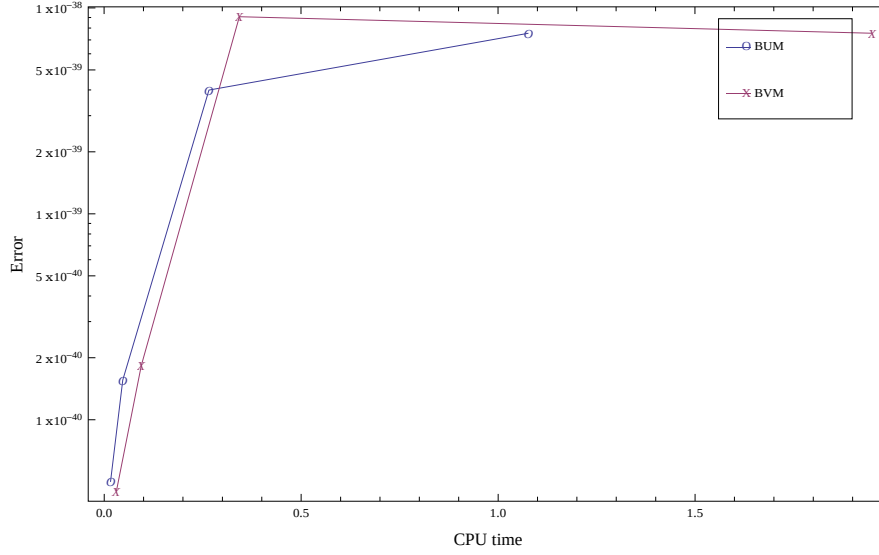


Figure 4.1.3: An efficiency graph of CPU time versus Errors for Problem 3

In the above efficiency curve, $N = 5, 10, 15, 20$.

Table 4.7: Numerical Results for Problem 4

x	Exact	BUM	BVM
0.1	0.9946538262680831	0.9946538262688414	0.9946538262681055
0.2	0.9771222065281362	0.9771222065307122	0.9771222065281914
0.3	0.9449011653032021	0.9449011653077824	0.9449011653032872
0.4	0.8950948185847621	0.8950948185909482	0.8950948185848665
0.5	0.8243606353500641	0.8243606353569659	0.8243606353501718
0.6	0.7288475201562036	0.7288475201627945	0.7288475201562975
0.7	0.6041258122411429	0.6041258122463661	0.6041258122412079
0.8	0.4451081856984936	0.4451081857017012	0.4451081856985219
0.9	0.2459603111156955	0.2459603111167922	0.2459603111156944
1.0	0.0	0.0	0.0

Table 4.8: Error of Methods for Problem 4

x	NMD	HBVP m =12	HBVP m =5	BUM h=0.1	BVM h=0.1
0.1	2.00×10^{-10}	1.97×10^{-13}	6.33×10^{-9}	1.61×10^{-14}	1.99×10^{-14}
0.2	7.00×10^{-10}	1.56×10^{-13}	7.50×10^{-9}	5.98×10^{-14}	4.97×10^{-14}
0.3	1.35×10^{-9}	1.83×10^{-13}	3.12×10^{-10}	1.06×10^{-13}	7.44×10^{-14}
0.4	2.00×10^{-9}	2.06×10^{-13}	3.91×10^{-9}	1.43×10^{-13}	8.58×10^{-14}
0.5	2.51×10^{-9}	2.02×10^{-13}	9.89×10^{-10}	1.48×10^{-13}	8.25×10^{-14}
0.6	2.72×10^{-9}	2.25×10^{-13}	1.75×10^{-9}	1.24×10^{-13}	6.53×10^{-14}
0.7	2.21×10^{-9}	2.04×10^{-13}	2.91×10^{-9}	8.51×10^{-14}	3.91×10^{-14}
0.8	1.80×10^{-9}	1.98×10^{-13}	1.01×10^{-8}	4.46×10^{-14}	1.18×10^{-14}
0.9	7.25×10^{-10}	2.18×10^{-13}	7.80×10^{-9}	1.26×10^{-14}	6.11×10^{-15}

HBVP is the method in Costabile and Napoli (2015); NMD is the method in Noor & Mohyud-Din (2007).

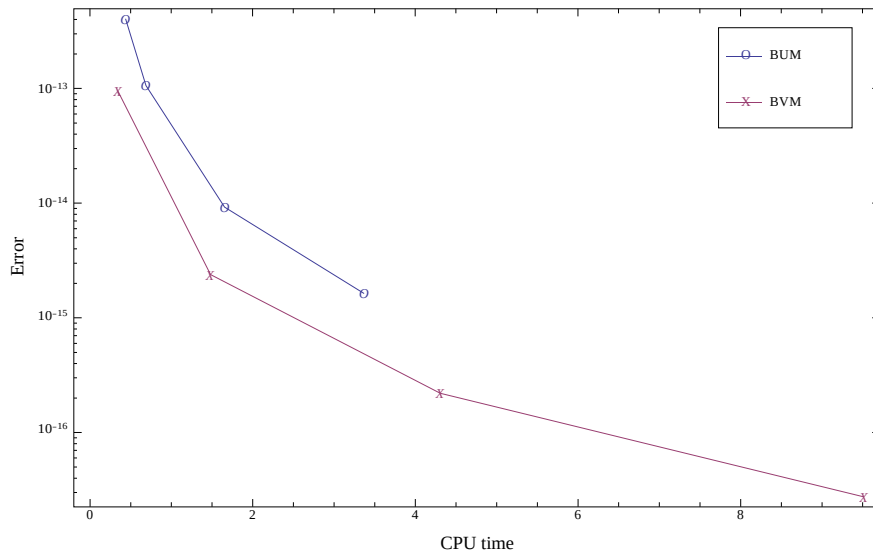


Figure 4.1.4: An efficiency graph of CPU time versus Errors for Problem 4

In the above efficiency curve, $N = 20, 40, 60, 80$.

Table 4.9: Numerical Results for Problem 5

x	Exact	BUM	BVM
0.	1.0	1.0	1.0
0.10680	1.038540815784233	1.03854081614349	1.038540815729688
0.21361	1.075651196733165	1.07565119743670	1.075651196641132
0.32042	1.111433534545771	1.11143353554372	1.111433534430435
0.42723	1.145979605823886	1.14597960703201	1.145979605697057
0.53404	1.179371990501769	1.17937199181936	1.179371990373226
0.64085	1.211685260994502	1.21168526232984	1.211685260872236
0.74766	1.242986985150365	1.24298698641722	1.242986985040755
0.85447	1.273338576932799	1.27333857805063	1.273338576840734
0.96128	1.302796021757989	1.30279602265718	1.302796021686963
1.06809	1.331410498017874	1.33141049864677	1.331410497970051

Table 4.10: Error of Methods for Problem 5

ADM in Duan & Rach (2011)		Method in Xu <i>et al.</i> (2016)		BUM	BVM	
$k_1 = 4$	3.5×10^{-4}	$k = 3, n = 6$	1.0×10^{-5}	$N = 5$	7.7×10^{-8}	7.7×10^{-8}
$k_1 = 5$	9.5×10^{-5}	$k = 3, n = 31$	4.0×10^{-7}	$N = 10$	1.2×10^{-8}	1.8×10^{-9}
$k_1 = 6$	2.3×10^{-5}	$k = 3, n = 81$	5.8×10^{-8}	$N = 15$	1.3×10^{-9}	1.2×10^{-10}

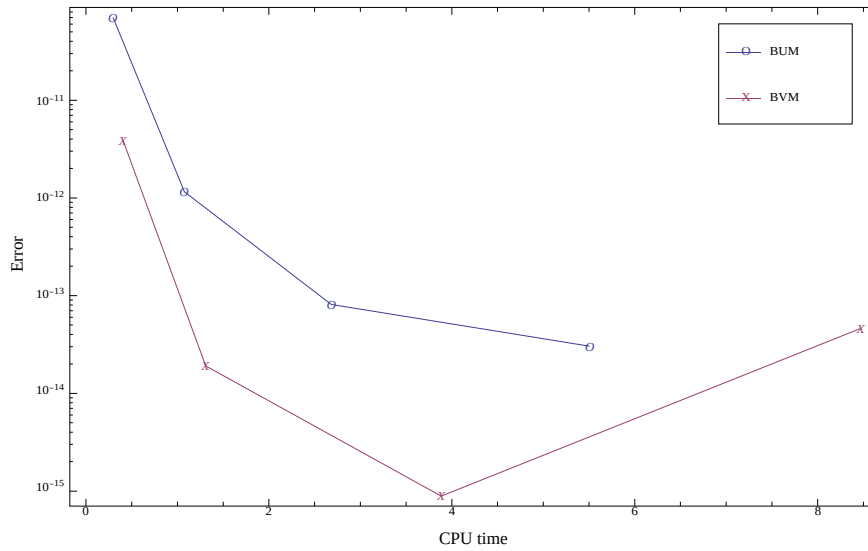


Figure 4.1.5: An efficiency graph of CPU time versus Errors for Problem 5

In the above efficiency curve, $N = 20, 40, 60, 80$,

4.1.5.2 Table of results for fifth order BVPs

Table 4.11: Numerical Results for Problem 6

x	Exact	BUM	BVM
0.0	1.0	1.0	1.0
0.1	1.105170918075647	1.105170918075685	1.105170918075647
0.2	1.221402758160169	1.221402758160418	1.221402758160159
0.3	1.349858807576003	1.349858807576687	1.349858807575971
0.4	1.491824697641270	1.491824697642543	1.491824697641206
0.5	1.648721270700128	1.648721270701961	1.648721270700027
0.6	1.822118800390508	1.822118800392602	1.822118800390379
0.7	2.013752707470476	2.013752707472340	2.013752707470345
0.8	2.225540928492468	2.225540928493671	2.225540928492369
0.9	2.459603111156954	2.459603111157359	2.459603111156910
1.0	2.718281828459045	2.718281828459045	2.718281828459045

Table 4.12: Error Methods for Problem 6

Methods	Maximum Absolute error (h=0.1)
BUM	2.098×10^{-12}
BVM	1.311×10^{-13}
RKSM	1.018×10^{-07}
EPCICM	1.121×10^{-05}
EPBICM	3.049×10^{-05}

BUM=Block Unification Method

BVM=Boundary Value Method

RKSM=Reproducing Kernel Space Method in Ghazala & Hamood (2013)

EPCICM=Embedded Perturbed Chebyshev Integral Collocation Method in Taiwo *et al.* (2015).

EPBICM=Embedded Perturbed Bernstein Integral Collocation Method in Taiwo *et al.* (2015).

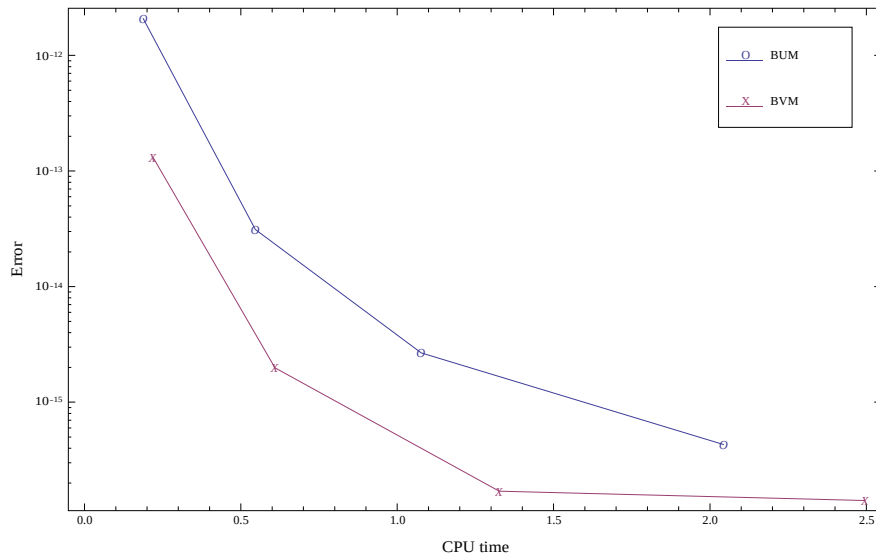


Figure 4.1.6: An efficiency graph of CPU time versus Errors for Problem 6

In the above efficiency curve, $N = 10, 20, 30, 40$.

Table 4.13: Numerical Results for Problem 7

x	Exact	BUM	BVM
0.0	0.0	0.0	0.0
0.1	0.095310179804324	0.095310179947166	0.095310179804324
0.2	0.182321556793954	0.182321557689514	0.182321556793954
0.3	0.262364264467491	0.262364266679648	0.262364264467491
0.4	0.336472236621212	0.336472240166730	0.336472236621212
0.5	0.405465108108164	0.405465112502574	0.405465108108164
0.6	0.470003629245735	0.470003633696629	0.470003629245735
0.7	0.530628251062170	0.530628254717318	0.530628251062170
0.8	0.587786664902119	0.587786667142482	0.587786664902119
0.9	0.641853886172394	0.641853886912650	0.641853886172394
1.0	0.693147180559945	0.693147180559945	0.693147180559945

Table 4.14: Error of Methods for Problem 7

Methods	Maximum Absolute error (h=0.1)
Method	Maximum Absolute Error
BUM (h=0.05)	4.5×10^{-9}
BVM (h=0.05)	1.1×10^{-10}
RKSM; Ghazala and Hamood (2013) (n=10)	9.2×10^{-6}
Method in Gamel (2007)	5.0×10^{-5}

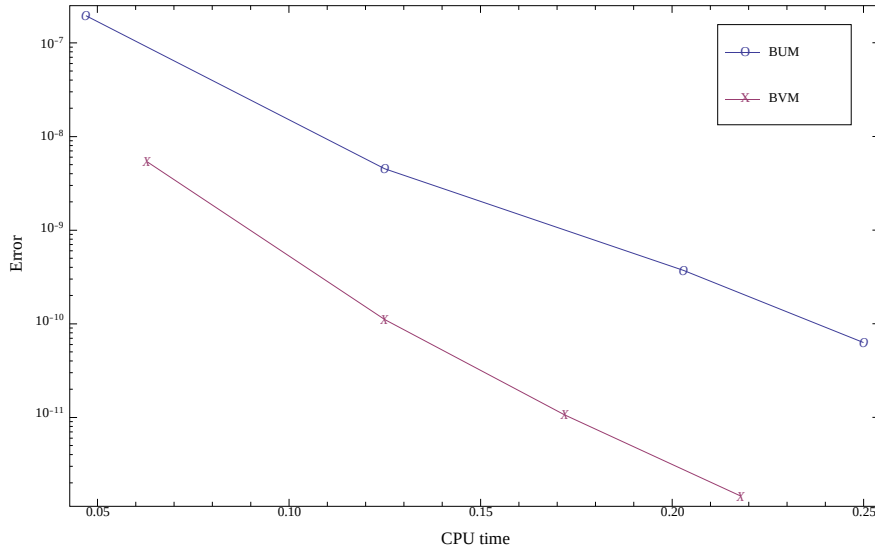


Figure 4.1.7: An efficiency graph of CPU time versus Errors for Problem 7

In the above efficiency curve, $N = 10, 20, 30, 40$.

Table 4.15: Numerical Results for Problem 8

x	Exact	BUM	BVM
0.0	0.0	0.0	0.0
0.1	0.11033298873020	0.110332988730615	0.110332988730252
0.2	0.24265526859492	0.242655268597562	0.242655268595079
0.3	0.39891055377848	0.398910553785825	0.398910553778852
0.4	0.58094390077056	0.580943900784372	0.580943900771183
0.5	0.79043908321361	0.790439083233675	0.790439083214438
0.6	1.02884566627209	1.028845666295080	1.028845666272985
0.7	1.29729511187526	1.297295111895870	1.297295111876047
0.8	1.59650534060025	1.596505340613688	1.596505340600749
0.9	1.92667330397271	1.926673303977363	1.926673303972886
1.0	2.28735528717884	2.287355287178842	2.287355287178842

Table 4.16: Maximum Error for Problem 8

N	BUM	BVM	N	Method in Pandey (2017)
15	5.50429×10^{-11}	8.71485×10^{-14}	16	6.25133×10^{-4}
30	7.96047×10^{-13}	1.39603×10^{-15}	32	1.62243×10^{-4}
60	1.21794×10^{-14}	1.00045×10^{-16}	64	4.45842×10^{-5}
120	2.30832×10^{-16}	1.01163×10^{-16}	128	1.14440×10^{-5}

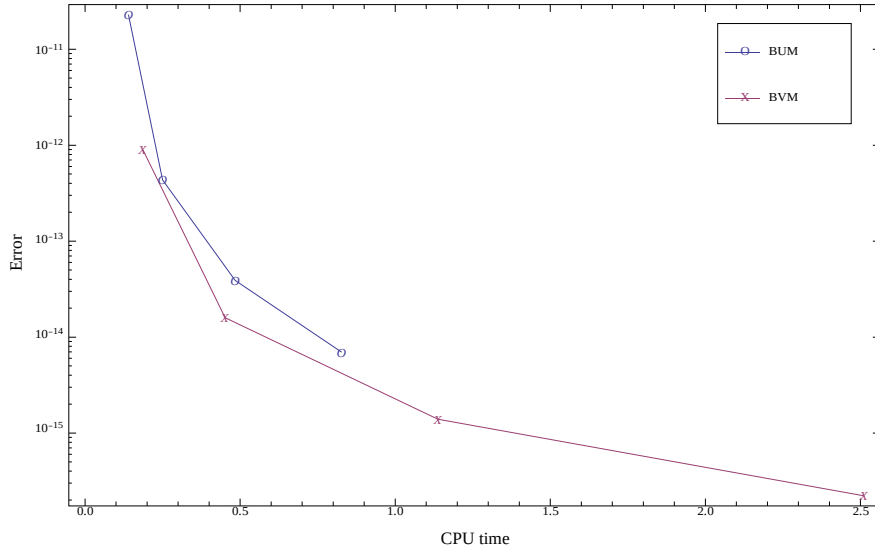


Figure 4.1.8: An efficiency graph of CPU time versus Errors for Problem 8

In the above efficiency curve, $N = 10, 20, 30, 40$.

Table 4.17: Numerical Results for Problem 9

x	Exact	BUM	BVM
0.0	1.609437912434100	1.609437912434100	1.609437912434100
0.1	1.629240539730274	1.629240539730279	1.629240539730279
0.2	1.648658625587381	1.648658625587382	1.648658625587380
0.3	1.667706820558076	1.667706820558081	1.667706820558074
0.4	1.686398953570228	1.686398953570243	1.686398953570227
0.5	1.704748092238425	1.704748092238449	1.704748092238423
0.6	1.722766597741103	1.722766597741131	1.722766597741102
0.7	1.740466174840504	1.740466174840534	1.740466174840503
0.8	1.757857917552373	1.757857917552391	1.757857917552373
0.9	1.774952350911673	1.774952350911685	1.774952350911673
1.0	1.791759469228055	1.791759469228053	1.791759469228055

Table 4.18: Maximum Error for Problem 9

N	BUM	BVM	N	Method in Pandey (2017)
15	9.35911×10^{-14}	4.80735×10^{-15}	16	1.97529×10^{-4}
30	1.42133×10^{-15}	1.43704×10^{-16}	32	4.38690×10^{-5}
60	2.20392×10^{-17}	4.08621×10^{-18}	64	1.14440×10^{-5}
120	4.16899×10^{-19}	3.58311×10^{-20}	128	2.50339×10^{-6}

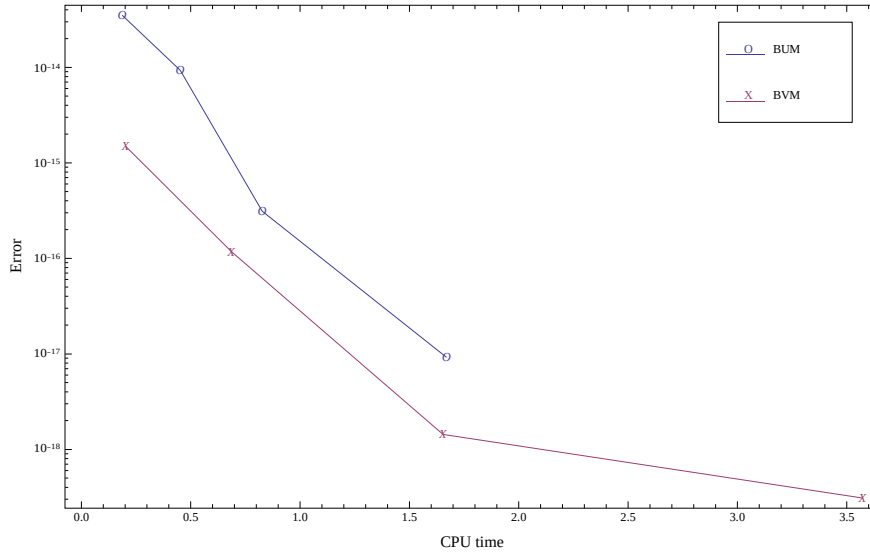


Figure 4.1.9: An efficiency graph of CPU time versus Errors for Problem 9

In the above efficiency curve, $N = 10, 20, 30, 40$.

Table 4.19: Numerical Results for Problem 10

x	Exact	BUM	BVM
0.0	0.0	0.0	0.0
0.1	0.099465382626808	0.099465382622882	0.099465382626553
0.2	0.195424441305627	0.195424441279433	0.195424441304642
0.3	0.283470349590960	0.283470349517462	0.283470349588504
0.4	0.358037927433904	0.358037927294443	0.358037927429534
0.5	0.412180317675032	0.412180317470109	0.412180317668941
0.6	0.437308512093722	0.437308511854703	0.437308512086833
0.7	0.422888068568800	0.422888068351518	0.422888068562571
0.8	0.356086548558794	0.356086548415590	0.356086548554708
0.9	0.221364280004125	0.221364279954324	0.221364280002820
1.0	0.0	0.0	0.0

Table 4.20: Maximum Error for Problem 10	
Method	Maximum Absolute error
BUM (h=0.001)	2.98×10^{-19}
BVM (h=0.001)	1.66×10^{-20}
GJPGM	6.35×10^{-17}

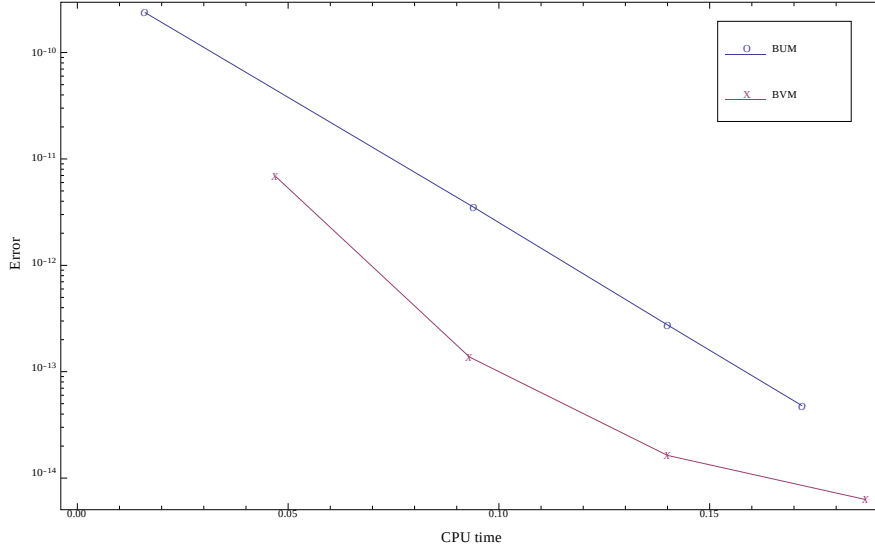


Figure 4.1.10: An efficiency graph of CPU time versus Errors for Problem 10

In the above efficiency curve, $N = 10, 20, 30, 40$.

4.1.5.3 Table of results for fourth order IBVPs in PDEs

Table 4.21: Numerical Results for Problem 11 at $t = 0$			
x	Exact	Approximate	Error
0	0	0	0
0.1	0.309016955747823	0.309016955724881	$2.29424812481227 \times 10^{-11}$
0.2	0.587785178819318	0.587785178848605	$2.92873503227042 \times 10^{-11}$
0.3	0.809016893247825	0.809016893187767	$6.00578475840052 \times 10^{-11}$
0.4	0.951056397413091	0.951056397460469	$4.73778793974588 \times 10^{-11}$
0.5	0.999999875000002	0.999999874925779	$7.42231831551976 \times 10^{-11}$
0.6	0.951056397413091	0.951056397460456	$4.73647787657682 \times 10^{-11}$
0.7	0.809016893247825	0.809016893187787	$6.00379745918644 \times 10^{-11}$
0.8	0.587785178819318	0.587785178848585	$2.92672552859585 \times 10^{-11}$
0.9	0.309016955747824	0.309016955724894	$2.29297136833395 \times 10^{-11}$
1.0	$1.22464664606 \times 10^{-16}$	$1.22464664606 \times 10^{-16}$	0

Table 4.22: Maximum Error for Problem 11						
Method	k	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
BUM	10	4.88E-10	8.56E-10	1.28E-9	1.39E-9	1.58E-9
	16	8.28E-10	1.50E-9	2.17E-9	2.43E-9	2.48E-9
Method in Rashidinia & Mohammadi, (2010)	10	2.91E-6	1.73E-6	1.60E-6	2.23E-6	2.60E-7
	16	4.47E-7	2.66E-7	1.37E-7	1.55E-7	1.57E-7

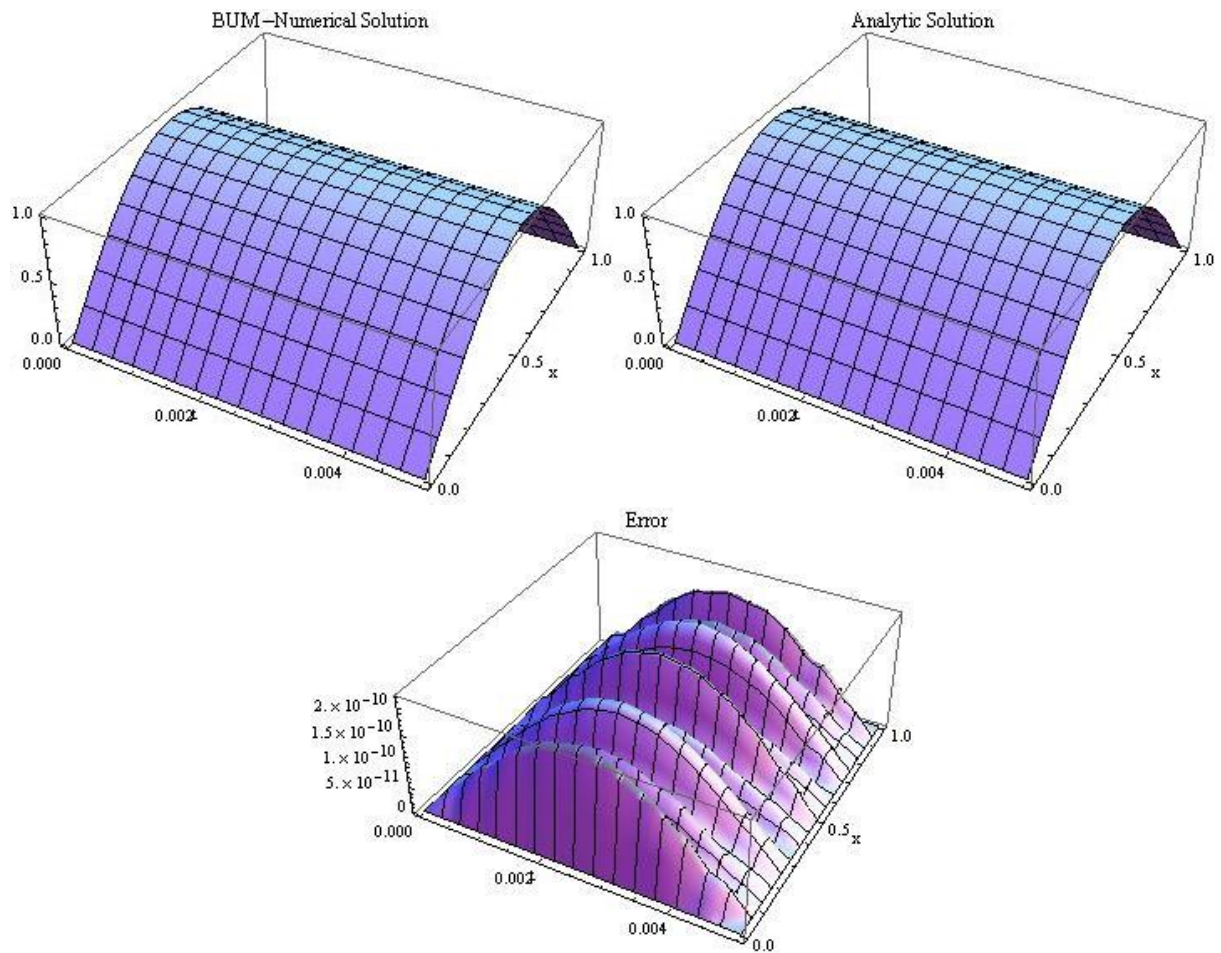


Figure 4.1.11: Surface plots for the numerical solution for Problem 11

Figure 4.1.11 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 11.

Table 4.23: Numerical Results for Problem 12 for $t = 0$

x	Exact	Approximate	Error
0.50	0.010002437457430	0.010002437711499	2.54068989805067E-10
0.55	0.010004027300724	0.010004027519762	2.19037874643612E-10
0.60	0.010006313226167	0.010006313409349	1.83182259291814E-10
0.65	0.010009502261557	0.010009502410985	1.49428013365704E-10
0.70	0.010013838934070	0.010013839053561	1.19490759301066E-10
0.75	0.010019608395203	0.010019608489015	9.38117639126545E-11
0.80	0.010027139545724	0.010027139617538	7.18140980709991E-11
0.85	0.010036808160620	0.010036808213003	5.23828064258324E-11
0.90	0.010049040014045	0.010049040048435	3.43898243215792E-11
0.95	0.010064314004267	0.010064314021325	1.70572948127167E-11
1.0	0.010083165278618	0.010083165278618	0.

Table 4.24: Maximum Error with for Problem 12

Method	Maximum absolute error	$h = 0.05, t = 0.01$
BUM	1.09E-10	
Method in Rashidinia	3.51E-8	
& Mohammadi, 2010.	[1ex]	

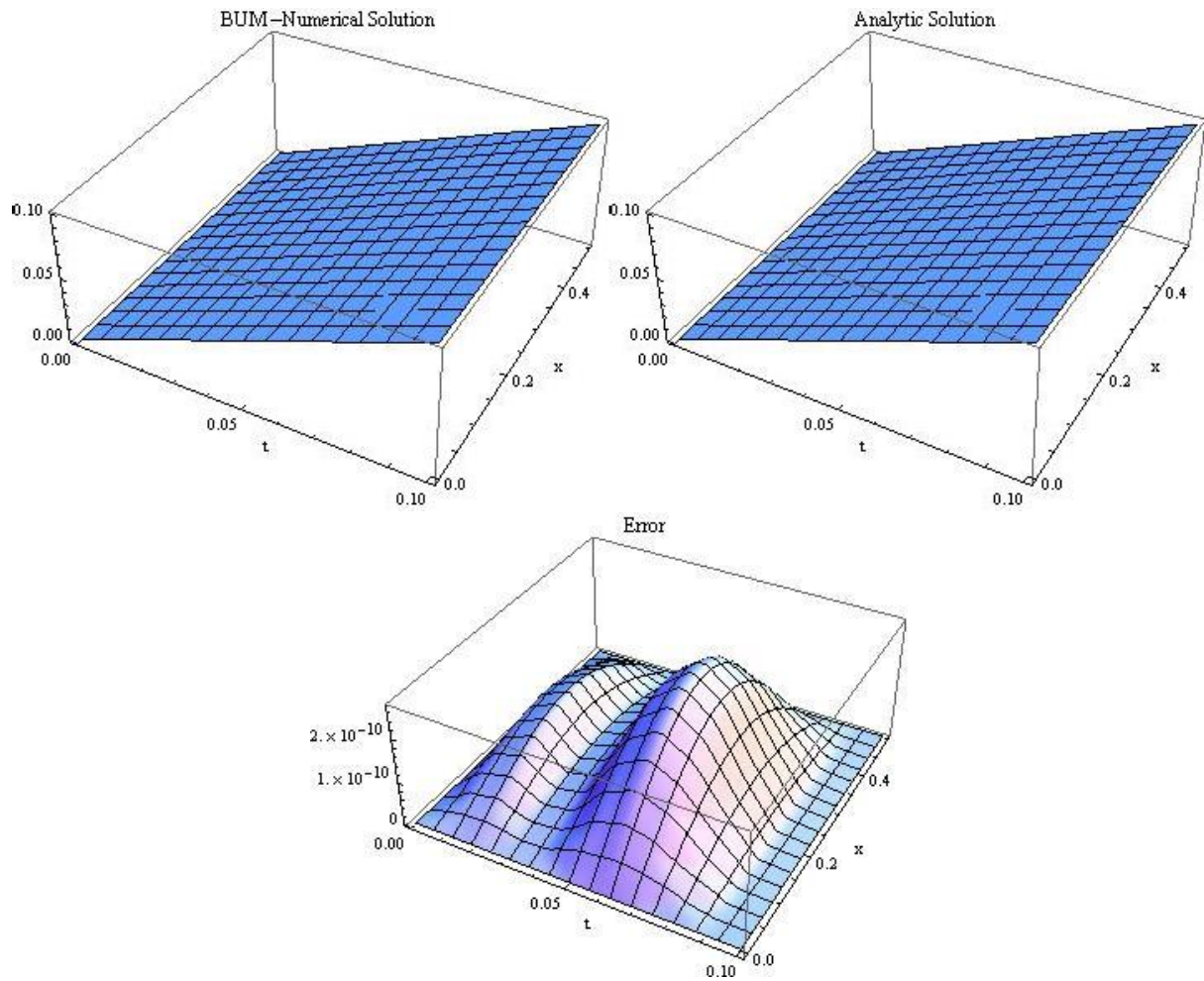


Figure 4.1.12: Surface plots for the numerical solution for Problem 12

Table 4.25: Numerical Results with $t = 0$ for Problem 13

x	Exact	Approximate	Error
0.1	0.000165752515231	0.000165752515231	0.
0.2	0.001324032463350	0.001324032464591	1.241245387723077E-12
0.3	0.004457450271551	0.004457450276171	4.619966735563974E-12
0.4	0.010528881468554	0.010528881453437	1.511686088806474E-11
0.5	0.020471845842386	0.020471845841486	9.004741396978488E-13
0.6	0.035181180199037	0.035181180205327	6.289448128971031E-12
0.7	0.055504097333941	0.055504097316726	1.721497250306569E-11
0.9	0.116091180938549	0.116091180906629	3.191993891427103E-11
1	0.157738348440023	0.157738348430269	9.754003160722391E-12

Table 4.26: Maximum Error with $h = 0.05$ for Problem 13

Method	k	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
BUM	10	0	1.24E-12	4.62E-12	1.51E-11	9.00E-13
	16	0	6.86E-12	1.68E-11	2.35E-12	8.30E-12
Method in Rashidinia and Mohammadi (2010)	10	8.35E-8	4.51E-8	8.25E-8	2.33E-8	4.52E-8
	16	8.42E-8	8.42E-8	5.32E-8	1.95E-8	2.89E-8

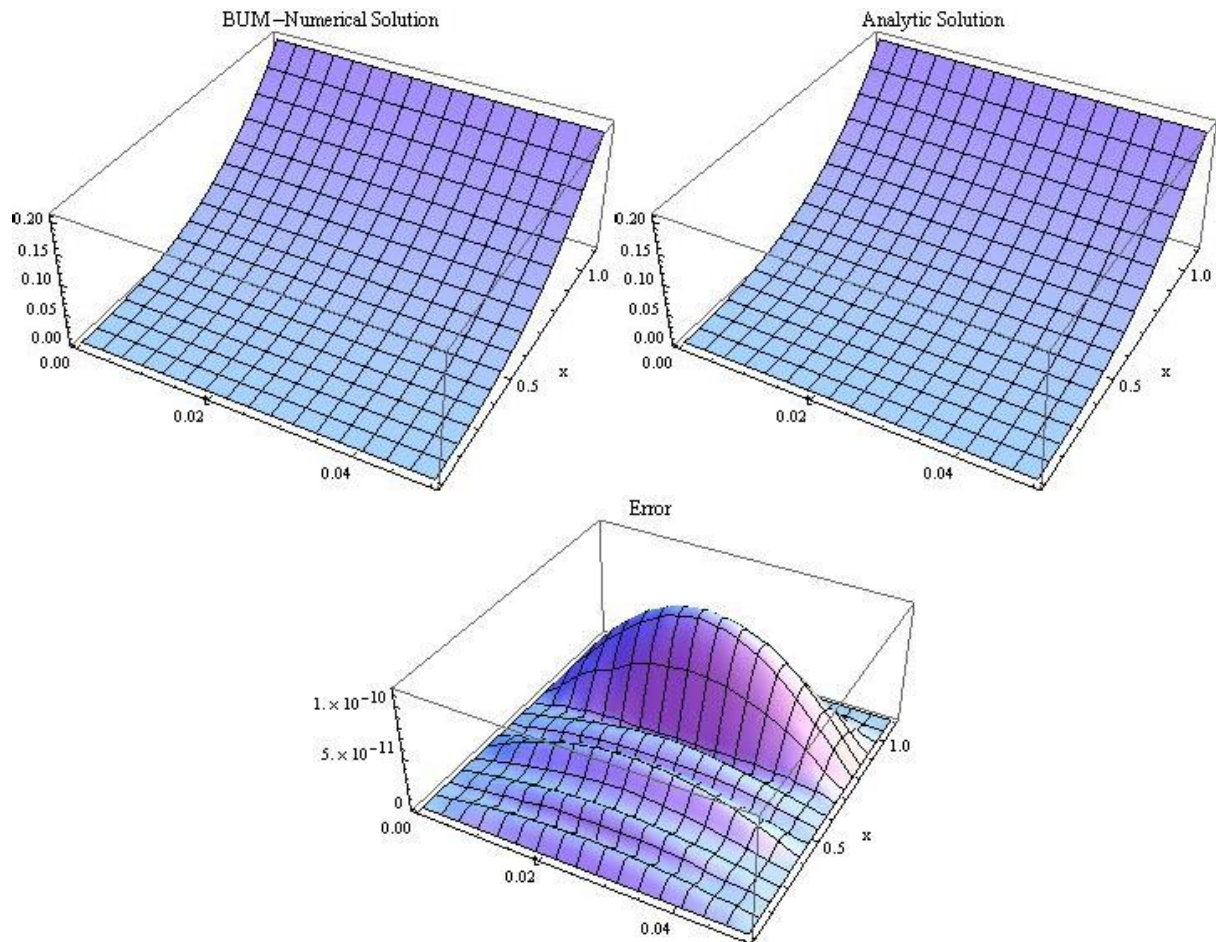


Figure 4.1.13: Surface plots for the numerical solution for Problem 13

Table 4.27: Numerical Results with $t = 0$ for Problem 14

x	Exact	Approximate	Error
0.1	1.190461309554812E-10	1.190461309554812E-10	0.
0.2	1.523790476230159E-8	1.523783265627949E-8	7.21060220980205E-14
0.3	2.6035388839963755E-7	2.603537652865388E-7	1.23113098773458E-13
0.4	1.950451809574604E-6	1.950451687610447E-6	1.21964157184599E-13
0.5	9.300478980896965E-6	9.300478934800345E-6	4.60966205798833E-14
0.6	0.000033325297715	0.000033325297799	8.44373060512014E-14
0.7	0.000098039607825	0.000098039608024	1.98767177305055E-13
0.9	0.000569393953930	0.000569393954078	1.48861717223780E-13
1	0.001190461309554	0.001190461309607	5.22316564999236E-14

Table 4.28: Maximum error with $h = 0.05$ for Problem 14

Method	k	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
BUM	10	0	7.21E-14	1.23E-13	1.22E-13	4.61 E-14
	16	0	4.40E-13	2.35E-13	4.34E-14	9.57E-14
Method in Rashidinia and Mohammadi (2010)	10	7.19E-10	2.26E-10	7.32E-10	6.91E-10	7.31E-10
	16	6.25E-10	2.22E-10	4.53E-10	4.41E-10	5.03E-10

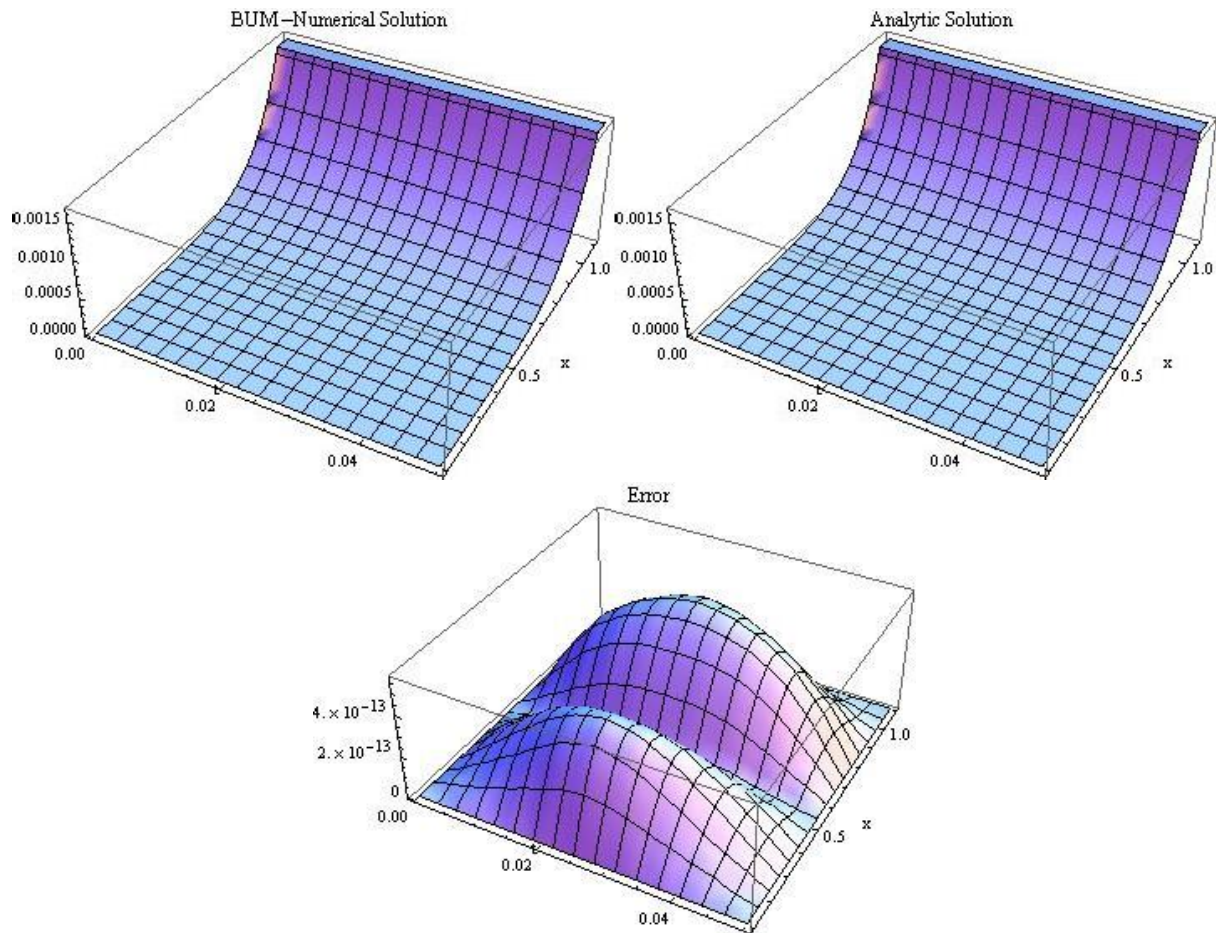


Figure 4.1.14: Surface plots for the numerical solution for Problem 14

Table 4.29: Numerical Results with $t = 0$ for Problem 15

x	Exact	Approximate	Error
0.1	1.708877984414899E-13	1.708877984414899E-13	0
0.2	1.626187988650137E-13	1.626187047633017E-13	9.4101711984771855E-20
0.3	1.547499235491294E-13	1.547497428773273E-13	1.8067180204290156E-19
0.4	1.472618110919623E-13	1.472615595514091E-13	2.5154055317655437E-19
0.5	1.401360369603024E-13	1.401357378286697E-13	2.9913163271624644E-19
0.6	1.333550681559634E-13	1.333547504007699E-13	3.1775519349533615E-19
0.7	1.269022200757641E-13	1.269019154383749E-13	3.0463738915634386E-19
0.8	1.201641402394919E-13	1.201638856826609E-13	2.5455683100524915E-19
0.9	1.149181453213176E-13	1.149179555268320E-13	1.897944856324631E-19
1	1.093574317791106E-13	1.093573319480806E-13	9.9831030002443655E-20

Table 4.30: Maximum Error for Problem 15

Time	Parameter	BUM	Method in Ref A	Method in Ref B
$t = 0.5$	$h = 1/40, v_0 = 30$	1.7274(-10)	7.8998(-07)	8.2943(-07)
$t = 1.0$	$h = 1/60, v_0 = 40$	1.7531(-12)	7.5071(-09)	7.3326(-09)
$t = 1.5$	$h = 1/80, v_0 = 50$	2.2691(-14)	5.7588(-11)	5.7588(-11)
$t = 2.0$	$h = 1/100, v_0 = 60$	1.6471(-16)	2.9068(-13)	2.9068(-13)

Ref A: Mohanty and Kaur (2016); Ref B: Siddiqi and Arshed (2014).

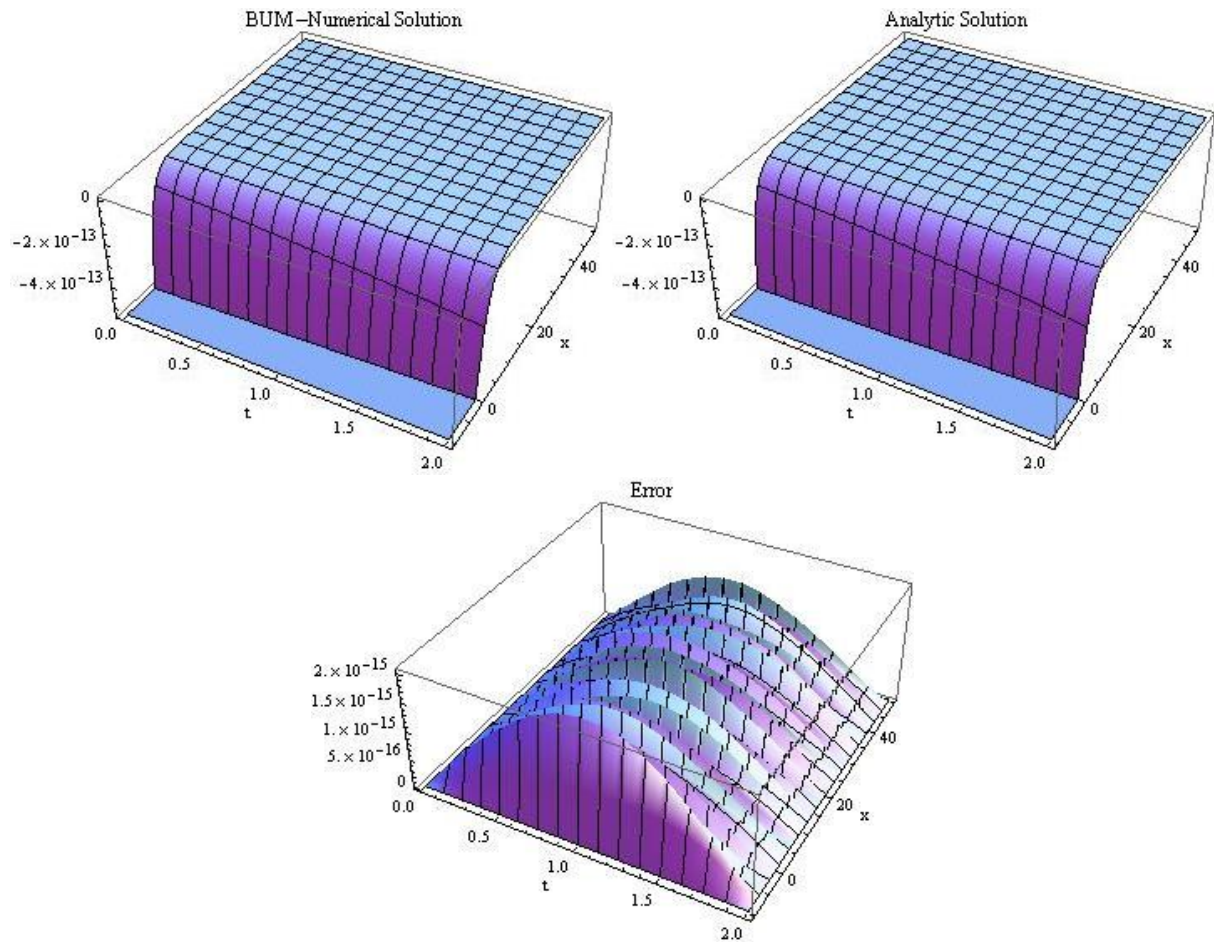


Figure 4.1.15: Surface plots for the numerical solution for Problem 15

4.1.5.4 Table of results for fifth order IBVPs in PDEs

Table 4.31: Numerical Results with $t = 1$ for Problem 16

x	Exact	Approximate	Error
0	-0.004999979166692	-0.004999979166692	0
0.15707	0.15149408848733	0.15149408870146	2.141388177889780E-10
0.23561	0.22858061645062	0.22858061751793	1.0673012751372113E-9
0.31415	0.30425786890274	0.30425787222302	3.3202844362811845E-9
0.47123	0.44952981081179	0.44952982708671	1.6274922243209033E-8
0.54977	0.51822885043904	0.51822888009699	2.9657957423268044E-8
0.62831	0.58373283687475	0.58373288662894	4.9754188879980185E-8
0.70685	0.64563791626026	0.64563799460913	7.8348871834776952E-8
0.86393	0.75714923383376	0.75714940263082	1.6879706343786438E-7
0.94247	0.80606796766763	0.80606820242075	2.3475312216358955E-7
1.02101	0.85001702443603	0.85001734178116	3.1734512684344907E-7

Table 4.32: Maximum Error for Problem 16 at $t = 1$

Method	N	L^1	L^2	L^∞
DGFEM	10	0.11E-1	0.12E-1	0.17
	20	0.39E-2	0.43E-2	0.61E-2
	40	0.12E-3	0.14E-3	0.19E-3
	80	0.36E-5	0.40E-5	0.57E-5
BUM	10	1.09E-5	4.76E-6	2.74E-6
	20	2.22E-6	6.74E-7	2.74E-7
	40	4.46E-7	9.53E-8	2.74E-8
	80	8.94E-8	1.34E-9	2.74E-9

DGFEM is Discontinuous Galerkin Finite Element Method developed by Cheng and Shu (2008).

where the p -norm is given as

$$L^p = ||y_i - y(x_i)||_p = \left(\sum_{i=1}^N |y_i - y(x_i)|^p \right)^{\frac{1}{p}}$$

$$L^\infty = ||y_i - y(x_i)||_\infty = \max_{1 \leq i \leq N} \{|y_i - y(x_i)|\}$$

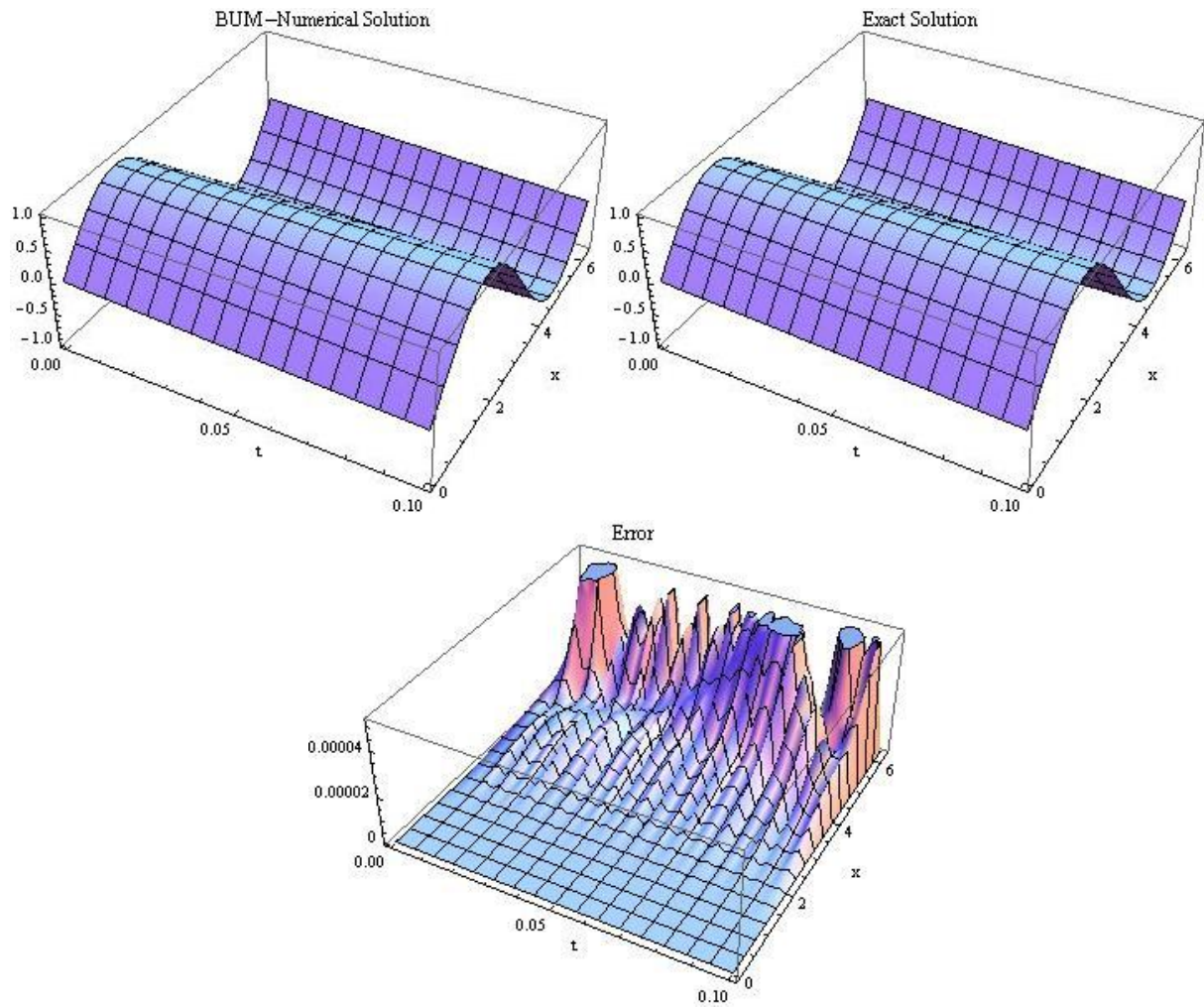


Figure 4.1.16: Surface plots for the numerical solution for Problem 16

Table 4.33: Numerical Results with $t = 1$ for Problem 17

x	Exact	Approximate	Error
0.1	1.094174283705210	1.094174283705210	0.
0.2	1.209249597657251	1.209249597640725	1.652589176615E-11
0.3	1.336427488025472	1.336427487787514	2.379574315369E-10
0.4	1.476980793882642	1.476980792811009	1.0716332266014E-9
0.5	1.632316219955379	1.632316217000374	2.9550042413006E-9
0.6	1.803988415397857	1.803988409268527	6.1293292663577E-9
0.7	1.993715533243082	1.993715522862050	1.0381031589901E-8
0.8	2.203396426255936	2.203396411511466	1.4744470533401E-8
0.9	2.435129651289874	2.435129634128278	1.7161596144177E-8
1	2.691234472349262	2.691234458263646	1.4085616228953E-8

Table 4.34: Maximum Error with $t = 0(0.1)0.5$ for Problem 17

t	E_6 in Goswami <i>et al.</i> (2017)	E_6 for BUM
0	0.00000000	000000000
0.1	2.00000000E-12	1.01100000E-14
0.2	8.00000000E-12	3.32110000E-14
0.3	1.03000000E-10	1.01111200E-14
0.4	7.66000000E-10	7.01254000E-14
0.5	3.61500000E-9	3.33210000E-14

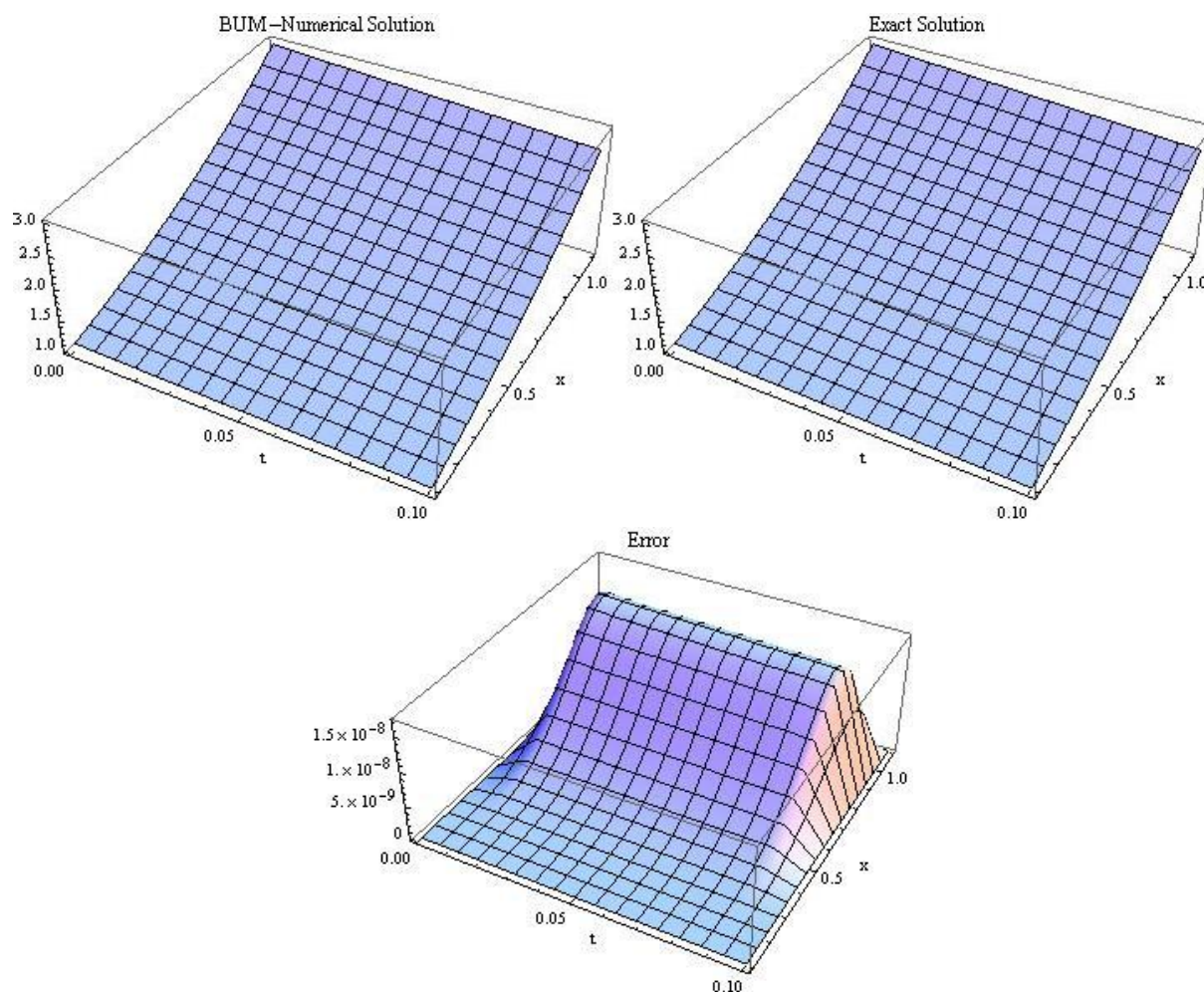


Figure 4.1.17: Surface plots for the numerical solution for Problem 17

Table 4.35: Numerical Results with $t = 1$ for Problem 18

x	Exact	Approximate	Error
0.1	3.9999940000000001E-10	3.9999940000000001E-10	0.
0.2	3.9999880000000001E-10	3.9999880000000001E-10	5.169878828456423E-26
0.3	3.9999820000000000E-10	3.9999820000000005E-10	4.135903062765138E-25
0.4	3.9999760000000001E-10	3.9999760000000010E-10	9.822769774067204E-25
0.5	3.9999700000000001E-10	3.9999700000000019E-10	1.809457589959748E-24
0.6	3.9999640000000001E-10	3.9999640000000029E-10	2.791734567366468E-24
0.7	3.9999580000000001E-10	3.9999580000000034E-10	3.308722450212111E-24
0.8	3.9999520000000001E-10	3.9999520000000035E-10	3.412120026781239E-24
0.9	3.9999460000000002E-10	3.9999460000000030E-10	2.843433355651032E-24
1	3.9999400000000002E-10	3.9999400000000019E-10	1.654361225106055E-24

Table 4.36: Maximum Error for Problem 18 at $t = 1$

Method	GA with RK	MQ with RK	IMQ with RK	CHB with BUM
L^2	1.5962E-20	5.0851E-11	1.5962E-20	5.18459E-24
L^∞	8.4703E-21	4.1421E-11	8.4703E-21	2.48154E-24

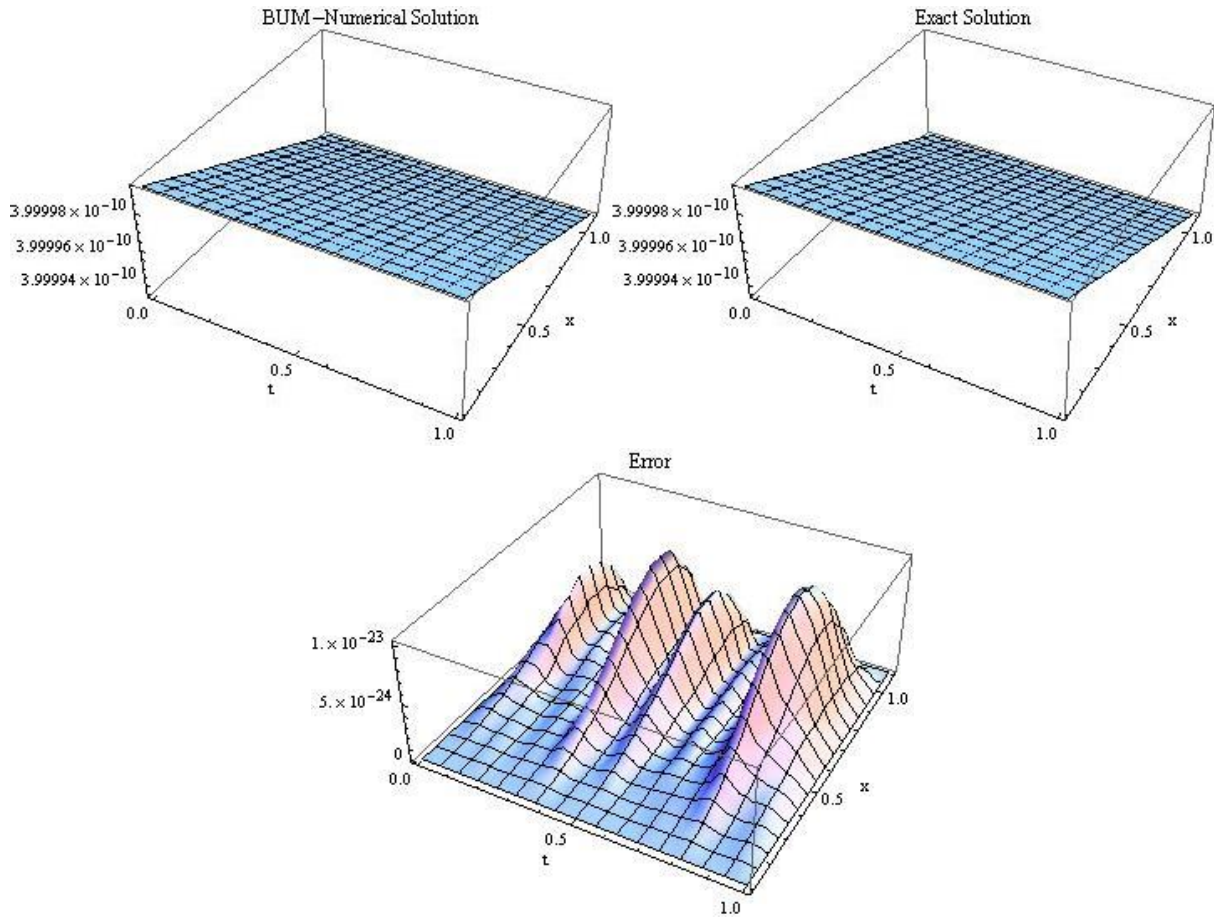


Figure 4.1.18: Surface plots for the numerical solution for Problem 18

Table 4.37: Numerical Results with $t = 1$ for Problem 19			
x	Exact	Approximate	Error
0.1	1.99999999998E-10	1.99999999998E-10	0.0000000000000000
0.2	1.99999999992E-10	1.99999999991E-10	1.292469707114105E-25
0.3	1.99999999982E-10	1.99999999981E-10	3.618915179919496E-25
0.4	1.99999999968E-10	1.99999999967E-10	6.720842476993354E-25
0.5	1.99999999950E-10	1.99999999949E-10	1.214921524687259E-24
0.6	1.99999999928E-10	1.99999999927E-10	1.757758801675183E-24
0.7	1.99999999902E-10	1.99999999901E-10	2.429843049374518E-24
0.8	1.99999999872E-10	1.99999999871E-10	2.998529720504725E-24
0.9	1.99999999838E-10	1.99999999837E-10	3.127776691216136E-24
1.0	1.99999999800E-10	1.99999999799E-10	2.403993655232236E-24

Table 4.38: Maximum Error for Problem 19 at $t = 1$				
Method	GA with RK	MQ with RK	IMQ with RK	CHB with BUM
L^2	1.9774E-21	5.9692E-12	1.9774E-21	5.96847E-24
L^∞	1.2705E-21	3.2896E-12	1.2705E-21	3.12778E-24

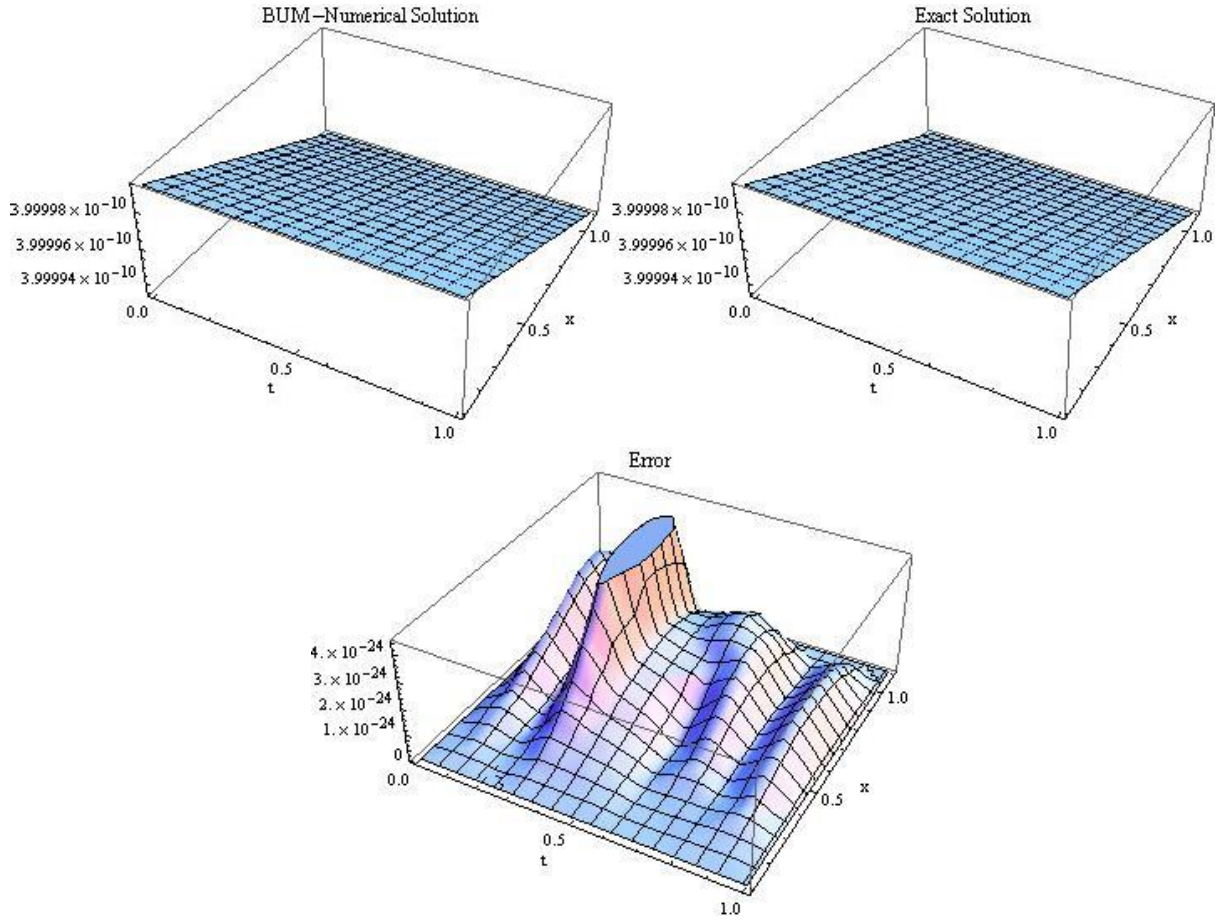


Figure 4.1.19: Surface plots for the numerical solution for Problem 19

Table 4.39: Numerical Results with $t = 1$ for Problem 20

x	Exact	Approximate	Error
0	0.0016666666666666666	0.0016666666666666666	0.
0.1	0.001666604167708631	0.001666604167708752	1.216474837528736E-16
0.2	0.001666416683333014	0.001666416683334928	1.914701036609401E-15
0.3	0.001666104251031847	0.001666104251041182	9.335414385969187E-15
0.4	0.001665666933274151	0.001665666933301932	2.778094594646507E-14
0.5	0.001665104817479392	0.001665104817541298	6.190577564457733E-14
0.6	0.001664418015980362	0.001664418016092191	1.118293825597138E-13
0.7	0.001663606665975485	0.001663606666143033	1.675480500867854E-13
0.8	0.001662670929470605	0.001662670929673168	2.025632266089427E-13
0.9	0.001661610993210292	0.001661610993379641	1.693491267357183E-13
1	0.001660427068598700	0.001660427068598700	0.

Table 4.40: Error of Methods for Problem 20

$t = 10^4$	DTM (x=-0.5)	BUM (x=-0.5)	DTM (x=1)	BUM (x=1)	DTM (x=2.5)	BUM (x=2.5)
0.0	1.0E-07	2.2E-12	2.0E-08	3.4E-14	1.0E-08	3.2E-14
1.0	3.0E-08	1.7E-12	2.0E-08	1.9E-14	2.0E-08	5.5E-14
2.0	5.0E-08	2.6E-12	7.0E-08	8.8E-14	3.0E-08	3.1E-14
3.0	5.0E-08	5.0E-12	1.0E-08	9.1E-14	1.0E-07	3.0E-14
4.0	1.0E-07	6.5E-12	1.0E-08	8.2E-14	3.0E-08	8.5E-14
5.0	3.0E-08	2.0E-12	2.0E-08	1.1E-14	4.0E-07	9.4E-14

DTM=Differential Transform Method developed in Modebei *et al.*, (2015).

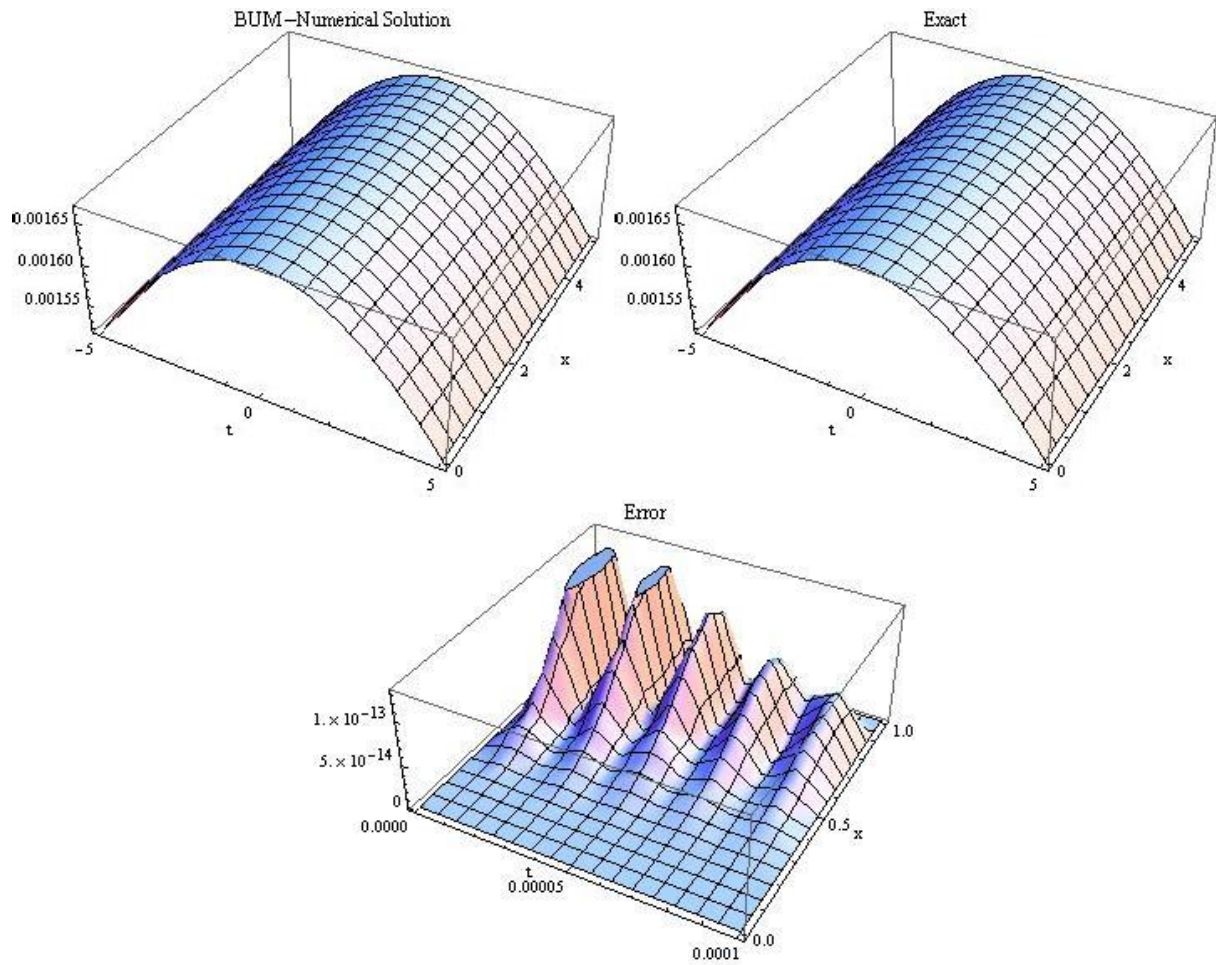


Figure 4.1.20: Surface plots for the numerical solution for Problem 20

4.2 Discussion of Results

Table 4.1 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM for Problem 1 for $N=10$, at the points $x = 0(0.1)1.0$.

Table 4.2 shows the errors obtained for applying the BUM and BVM compared to the errors obtained in Noor and Mohyud-Din (2007), and Costabile and Napoli (2015). It can be seen that the BVM performed better than the BUM. Overall, the BVM and BUM performed better compared to methods of Noor and Mohyud-Din (2007), and Costabile and Napoli (2015).

The efficiency graph in Figure 4.1.1 shows the CPU time versus the error for the BVM and the BUM for Problem 1. The CPU time is the time needed for the method (BVM or BUM) to solve an underlining problem. The BUM and BVM curves are inter-twined, showing that they both almost have the same CPU time at three points. The BVM line appears extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is more costly in terms of CPU time while the BUM is less time consuming.

Table 4.3 shows the comparison of the exact solution and the numerical solutions obtained using BUM and

BVM at the points $x = 0(0.1)1.0$ for Problem 2. Table 4.4 shows the errors obtained for applying the BUM and BVM compared to the errors obtained in Costabile and Napoli (2015).

$Er1^{method}$, $Er2^{method}$ and $Er3^{method}$, are different errors obtained for $c = -1/2, 5$ and 20 respectively. It can be seen that the BVM performed better than the BUM. Overall, the BVM and BUM performed better compared to the method of Costabile and Napoli (2015). The efficiency graph in Figure 4.1.2 shows the CPU time versus the error for the BVM and the BUM for $c = -1/2$. The BUM and BVM curves are inter twined showing that they both almost have the same CPU time at three points. The BVM curve appears extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is more costly in terms of CPU time while the BUM is less time consuming.

Table 4.5 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 3. Table 4.6 shows the maximum errors obtained for applying the BUM and BVM compared to the errors obtained with different methods in Tatari and Dehghan (2006), Duan and Rach (2011) and Xu, *et al* (2016). The BVM and BUM were implemented for N divisible by $k = 5$, where N is the number of subintervals in the interval of integration $[a, b]$, while k is the step length. It can be seen that the BVM performed better than the BUM. Overall, the BVM and BUM performed better compared to the methods in the above cited literature. On the other hand, the efficiency graph in Fig 4.1.3 shows the CPU time versus the error for the BVM and the BUM. As seen earlier, the BVM curve appears extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, The BVM is more costly in terms of CPU time while the BUM is less time consuming.

Table 4.7 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 4. Table 4.8 show the errors obtained in Problem 4 with $Er = |y_{app} - y_{ex}|$, for $h = 0.1$, as compared with methods in Costabile and Napoli (2015) and Noor & Mohyud-Din (2007). This shows the superiority of the BVM developed over the methods in the cited literatures. The efficiency graph in Figure 4.1.4 shows the CPU time versus the error for the BVM and the BUM. The BUM and BVM curves are inter twined showing that they both almost have the same CPU time at three points. The BVM curve appears below (showing it has minimal error compared to BUM), extended to the right showing that it took more time to arrive at the third and last points. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is more costly in terms of CPU time while the BUM is less time consuming.

Table 4.9 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at different points of x for Problem 5. Table 4.10 shows the errors obtained for applying the BUM and BVM compared to the errors obtained in Duan & Rach (2011) and Xu *et al.* (2016). It can be seen that the BVM performed slightly better than the BUM. Overall, the BVM and BUM performed better compared to methods in Duan & Rach (2011) and Xu *et al.* (2016). The efficiency graph in Figure 4.1.5 shows the CPU time versus the error for the BVM and the BUM. The BVM curve appears below the BUM (showing its superiority in terms of the errors) and extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is more costly in terms of CPU time while the

BUM is less time consuming.

Table 4.11 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 6. Table 4.12 presents the maximum error obtained using the BUM and BVM, method in Ghazala and Hamood (2013) and methods in Taiwo *et al.* (2015). The BVM performed better than the BUM but in general, both methods performed better are more accurate in terms of the errors obtained when compared with the other methods mention in the cited literature. The efficiency graph in Figure 4.1.5 shows the CPU time versus the the error for the BVM and the BUM. The BVM curve appears below the BUM (showing its superiority in terms of the errors) and extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is more costly in terms of CPU time while the BUM is less time consuming.

Table 4.13 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 7. Table 4.14 presents the maximum absolute error obtained using the BUM, BVM and methods in Ghazala and Hamood (2013) and Gamel (2007). The errors obtained show that the BVM, apart from being superior to BUM, both methods are superior compared to the those in Ghazala and Hamood (2013) and Gamel (2007). The efficiency graph in Figure 4.1.7 shows the CPU time versus the the error for the BVM and the BUM. The BVM curve appears below the BUM (showing its superiority in terms of the errors) and both show almost the same CPU time with the BUM a little more costly than the BVM.

Table 4.15 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 8. Unlike the method in Pandey (2017), we recall that the Block Unification Method (BUM) and Boundary Value Method (BVM), are applied on block of intervals of the form $[x_n, x_{n+5}]$. Here $n = 0, 5, \dots, N - 5$, where N , is the number of subintervals and must be a multiple of 5 in order to reach the final point $x_N = b$. This can be seen in Table 4.16. The BVM is more superior than the BUM but both methods are superior to the method in Pandey (2017) in terms of the errors obtained. The efficiency graph in Figure 4.1.8 shows the CPU time versus the the error for the BVM and the BUM. The BVM curve appears below the BUM (showing its superiority in terms of the errors) and both show almost the same CPU time with the BUM a little costly than the BVM.

Table 4.17 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 9. Similarly as in Problem 8, Table 4.17 shows the solution obtained for Problem 9. In Table 4.18, at different values of N as above, it shows the maximum absolute errors obtained, in comparison with the BUM, BVM and the method of Pandey (2017). Again, this shows the superiority of the BUM and the BVM to method Pandey (2017), in in terms of the errors obtained. The efficiency graph in Figure 4.1.9 shows the CPU time versus the the error for the BVM and the BUM. The BVM curve appears below the BUM (showing its superiority in terms of the errors) and extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is costly in terms of CPU time while the BUM is less time consuming.

Table 4.19 shows the comparison of the exact solution and the numerical solutions obtained using BUM and BVM at the points $x = 0(0.1)1.0$ for Problem 10. The method used by Doha *et al.* (2015) is a Nonsymmetric Generalized Jacobi Petrov-Galerkin method (GJPGM) uses P_N , which is the space of all polynomials of degree less than or equal to N . The result was obtained for $N = 15$. It can be seen that the BUM and BVM performed well for very small values of h in Table 4.20. The efficiency graph in Figure 4.1.10 shows the CPU time versus the error for the BVM and the BUM. The BVM curve appears below the BUM (showing its superiority in terms of the errors) and extended to the right showing that it took more time to arrive at the last point. This shows clearly that, despite the superiority of the BVM over the BUM, the BVM is costly in terms of CPU time while the BUM is less time consuming.

Table 4.21 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at the points $x = 0(0.1)1.0$ for Problem 11. Table 4.22, at different values of k and points of x , shows the maximum absolute errors obtained, in comparison with the BUM and the method in Rashidinia and Mohammadi, 2010. This shows the superiority of the BUM. Figure 4.1.11 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 11.

Table 4.23 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at the points $x = 0(0.1)1.0$ for Problem 12. Table 4.24, at $t = 0.01$ and a step-size $h = 0.01$, shows the maximum absolute errors obtained, in comparison with the BUM and the method in Rashidinia and Mohammadi, (2010). This shows the superiority of the BUM. Figure 4.1.12 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 12.

Table 4.25 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at the points $x = 0(0.1)1.0$ for Problem 13. Table 4.26, for different values of k and at $x=0.1(0.1)0.5$, shows the maximum absolute errors obtained, in comparison with the BUM and the method in Rashidinia and Mohammadi, 2010. This shows the superiority of the BUM. Figure 4.1.13 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 13.

Table 4.27 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at the points $x = 0(0.1)1.0$ for Problem 14. Table 4.28, for different values of k and at $x = 0.1(0.1)0.5$, shows the maximum absolute errors obtained, in comparison with the BUM and the method in Rashidinia and Mohammadi, 2010. This shows the superiority of the BUM. Figure 4.1.14 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 14.

Table 4.29 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at the points $x = 0(0.1)1.0$ for Problem 15. Table 4.30, for different values of t and h , shows the maximum absolute errors obtained, in comparison with the BUM and the method in Mohanty & Kaur (2016) and Siddiqi & Arshed (2014). This shows the superiority of the BUM over both methods. Figure 4.1.15 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 15.

Table 4.31 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at different the points of x for Problem 16. Table 4.32, for different subinterval N , shows the maximum errors L^∞ and the L^p with $p = 1, 2 - norm$ obtained, in comparison with the BUM and the method in Cheng and Shu (2008). This shows the superiority of the BUM. Figure 4.1.16 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 16.

Table 4.33 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at different the points of x for Problem 17. Table 4.33, for different values of t and $E(6)$, shows the maximum absolute errors obtained, in comparison with the BUM and the method in Goswami *et al.* (2017). This shows the superiority of the BUM. Here $E(n) = \sum_{i=1}^n |y_i - y(x_i)|$. Figure 4.1.17 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 17.

Table 4.35 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at different the points of x for Problem 18.

Figure 4.1.18 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 18.

Table 4.37 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the error at different the points of x for Problem 19.

Figure 4.1.19 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 19.

Mohyud-Din *et al* (2012) presented a numerical solution of Problems 18 and 19 using a meshless method of lines. This method uses Meshless, Multiquadric (MQ), Inverse multiquadric (IMQ) and Gaussian (GA) as Radial basis function (RBF) for spatial derivatives and Runge-Kutta method as a time integrator.

In this work, a constant mesh method of line was applied. the Block Unification Method (BUM) derived in this work was used as the space derivative integrator with the Chebyshev (CHB) polynomials as basis function while the time variable was discretized using the central difference approximation. The constant step-size used for Table 4.35, 4.36, 4.37 and 4.38 is $h = 0.1$.

Table 4.39 shows the comparison of the exact solution, the numerical solutions obtained using BUM and the

error at different the points of x for Problem 20. Table 4.40, for different values of t and at $x = -0.5, 1, 2.5$, shows the maximum absolute errors obtained, in comparison with the BUM and the method in Modebei *et al.*, (2015). This shows the superiority of the BUM over the DTM. Figure 4.1.20 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Problem 20.

The proposed methods BVM and BUM performed well in terms of accuracy than the existing methods cited in the literature. On the other hand, comparing the BVM and the BUM, the BVM is preferable to the BUM if only the errors are to be considered. Suppose the CPU time is considered here, then the BUM is preferred to the BVM since the BVM is costly in terms of implementation and execution for the problems under consideration. This can be seen in Figures 4.1.1-4.1.10. The BVM was not discussed for Problems (4.1.11)-(4.1.38).

CHAPTER FIVE: CONCLUDING REMARKS

5.1 Summary

In this thesis, two new classes of continuous block schemes for solving fourth and fifth order Initial-Boundary Value Problems (IBVPs) in differential equations were developed. These methods were constructed via the interpolation/collocation approach. The Shifted Chebyshev polynomial was used as basis function for the trial solution over the interval $[x_n, x_{n+k}]$, where k is the step-number. The derived schemes do not need any other external methods to give or guess the starting values. In other words, the derived block methods are self-starting and hence have no need for predictors. From the types of problems discussed and the table of results seen in the previous section, it can be clearly seen that the proposed methods are robust as they have wide scope of application viz-a-viz linear and nonlinear in ODEs and PDEs.

Application problems such as Cantilever beam construction and bridges in engineering, magnetohydrodynamic (MHD) problems, models arising from the propagation of shallow water waves over a flat surface, fully nonlinear Korteweg-de Vries (KdV)-like equation, the good Boussinesq equations, the Kaup-Kupershmidt equations and an extended KdV5 equation which models nonlinear dispersive waves, just to mention few. The tables of results show that the developed methods can be adopted for solving fourth and fifth order IBVP in differential equations.

5.2 Conclusion

Two classes of methods namely, the Block Unification Method (BUM) and the Boundary Value Method (BVM) have been developed. These methods consist of continuous linear multistep schemes assembled in block fashion to solve fourth and fifth order BVPs in ODEs and fourth and fifth order IBVPs in PDEs. The problems were solved without first reducing ODEs to an equivalent first order system. It is noteworthy that the PDEs were first semi-discretized into system of ODEs of equivalent order by the approach of the Method of Lines (MOLs). The categories of problems solved using these methods showed their robustness. The derived methods were shown to satisfy all necessary and sufficient properties of numerical methods for solution of ODEs. These properties include (for BVPs); consistency and convergence. The Proof of convergence was also explicitly shown.

All computations were carried out by computer programs written in Wolfram Mathematica 8.0, on the Windows 7 operating system; 2.10 GHz Processor speed and 3.00 GB internal memory (RAM). The computer programs contain no subroutines written in other files, which means the general computing time is less.

5.3 Contribution to Knowledge

The following contributions are made to the body of knowledge;

1. a new class of continuous implicit five-step methods for the direct solution of general fourth order and fifth order BVPs of ODE were developed;
2. a new class of continuous implicit five step methods for the solution of general fourth and fifth IBVPs of PDE were developed;
3. the derived schemes performed favourably well when compared to other methods in the literature.
4. very accurate and highly efficient computer codes were written for the implementation of the new methods;
5. the new class methods have a wide scope of application viz-a-viz linear, nonlinear, singular, nonsingular, homogenous, nonhomogenous, constant coefficients, variable coefficients, oscillatory problems.

5.4 Recommendation

The proposed methods reported in this thesis have been confirmed to perform better in terms of accuracy than existing methods in literature. Hence, they are recommended for solution of problems arising from physical phenomena leading to fourth and fifth order BVPs in ordinary differential equations and also fourth and fifth order IBVPs in partial differential equations.

5.5 Suggestion for Further Work

Methods derived in this thesis are for fourth and fifth order boundary Value problems in ordinary differential equations and fourth and fifth order initial-boundary value problems in partial differential equations. Furthermore, the Chebyshev polynomials have been used as basis function for the trial solution. It is therefore suggested that other orthogonal polynomials (conventional or constructed) may be used as basis functions for the trial solution. This may further improve the results obtained.

It can also be seen that the BVM was not used for any of the PDEs; it is also suggested that appropriate BVM may be derived which satisfies all numerical properties including stability and convergence for PDEs.

Finally, Hybrid methods can also be derived for the aforementioned problems. This may as well improve the results obtained in this thesis.

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Appendices

Appendix I Computer Program for BUM for solving 4th order ODEs

■ Solve $y^{(iv)} = f(x, y, y', y'', y''')$, $a_0 \leq x \leq b_0$,

```
In[1]:= a = a0; b = b0; N = IntegerModulo k; (*input integration intervals [a0,b0],
      N: number of integration subintervals; k: step-number*)
      h = (b0 - a0) / N; (*calculate h: step-size*)
      x0 = starting value;

In[4]:= y0 = y - initial;
      y'N = Finalyprime; y''0 = YprimeprimeInitial;
      y'''N = YprimeprimeFinal;
      (*the above are the boundary conditions (for fourth order problem)*)

In[7]:= x[n_] := xn = x0 + n * h (*points of intergration and evaluation*)

In[8]:= Y[n_] := Yvector1; (*generate all vectors for y*)

In[9]:= Yp[n_] := Yvector2; (*generate all vectors for y'*)

In[10]:= Ypp[n_] := Yvector3; (*generate all vectors for y''*)

In[11]:= Yppp[n_] := Yvector4; (*generate all vectors for y'''*)

In[12]:= f := (*Problem to be solved in the form y(iv)=f(x,y,y',y'',y''')*);
```

This the Derived Block Method

```
In[12]:= BlockDerived = {};

In[13]:= Equations = Table[BlockDerived, {i, 0, N - 4, 4}] // Flatten;
      (*BlockDerived for i=0, step of 4 to N-4*)
```

Variables

```
In[14]:= YInitVectorSolution = Join[YVector1, YVector2, YVector3, YVector4];

In[15]:= MakeGuessPair[p_] := {p, 1}
      InitialRootGuesses = Map[MakeGuessPair, YInitVectorSolution];

In[17]:= time = Timing[sol = FindRoot[Equations, InitialRootGuesses, WorkingPrecision -> 30]];
```

Output of solutions in the table form

```
In[18]:= q0 = Table[y1 /. sol, {i, 0, N}];

In[19]:= q1 = Table[yp1 /. sol, {i, 0, N}];
```

Exact Solution

```
In[20]:= YExact[x_] := ExactSolution (*if known*);  
In[21]:= UVectors = Table[YExact[x[i]], {i, 0, N}];  
In[22]:= ye = Table[YExact[i], {i, a0, b0, h}];
```

Numerical Results

```
In[23]:= Error = Abs[UVectors - q0];  
In[24]:= time[[1]] (*return CPU time for finding solution*)  
In[25]:= NormInf = Max[Error] (*return maximum error also known as  $L^\infty$  *)  
In[26]:= Norm1 = Norm[UVectors1 - q0, 1] (*return maximum error also known as  $L^1$  *)  
In[27]:= Norm2 = Norm[UVectors1 - q0, 2] (*return maximum error also known as  $L^2$  *)  
In[28]:= RESULT = Table[{N[x[i - 1]], q0[[i]], N[ye[[i]]], q0[[i]] - ye[[i]]}, {i, 1, N + 1}];  
In[29]:= Prepend[RESULT, {"x", "yapprox", "yExact", "yError"}] // MatrixForm  
Prepend[RESULT, {"x", "yapprox", "yExact", "yError"}]
```

Appendix II Computer Program for BVM for solving 4th order ODEs

Computer programme for implementation of the BVM is similar to the above code for BUM but the input lines In[12] and In[13] are replaced with the code below;

Main Methods

```
In[12]:= EquationMain0 = {};  
EquationMain1 = Table[EquationMain0, {i, 0, N - 4}] // Flatten;  
(*BlockDerived for i=0, step of 4 to N-4*)
```

Additional methods

```
In[14]:= EquationAdditional2 = {};  
In[15]:= EquationBlock = Join[EquationMain1, EquationAdditional2];
```

Appendix III Computer Program for BUM for solving 4th order PDEs

■ Code for solving PDE using the BUM

Input Variables

```
In[1]:= N = 10; M = 10;

In[2]:= L3 = 0; L4 = 1; L1 = 0; L2 = 1; h = Δx = (L2 - L1) / N; δ = Δt = (L4 - L3) / M; x0 = 0.1; t0 = 0.0;

In[3]:= t[n_] := (tn = t0 + n * δ);
(*This assigns a vector x to the expression on the right*)
SystemOfEqs = Map[t, Range[0, M]];
```

■ Generating the mesh on the y-axis

```
In[5]:= x[n_] := (xn = x0 + n * h);
(*This assigns a vector x to the expression on the right*)
SystemOfEqs = Map[x, Range[0, N]];
```

■ Exact solution of the problem

```
In[7]:= UB[t_, x_] := Exact Solution;
```

■ Boundary conditions of the PDEs

```
In[8]:= Table[u1,0 = UB[t[i], x0], {i, 0, M}];

In[9]:= Table[u1,N = UB[t[i], xm], {i, 0, M}];

In[10]:= Table[up1,0 = D[UB[t[i], x], {x, 1}] /. x -> x0, {i, 0, M}];

In[11]:= Table[upp1,0 = D[UB[t[i], x], {x, 2}] /. x -> x0, {i, 0, M}];
```

■ Generating system of fourth order ODEs

```
In[12]:= Y = u1,n; YVector = {Y1,n, 1, 1, M, n, 1, N} // Flatten;

In[13]:= Yp = up1,n; YpVector = {Yp1,n, 1, 1, M, n, 1, N} // Flatten;

In[14]:= Ypp = upp1,n; YppVector = {Ypp1,n, 1, 1, M, n, 1, N} // Flatten;

In[15]:= Yppp = ut1,n; YpppVector = {Yppp1,n, 1, 1, M, n, 1, N} // Flatten;
```

- Problem to be solved in the form $f_{m,n} = (x_n, y_{m,n}, y'_{m,n}, y''_{m,n}, \dots)$

```
In[16]:= f_m,n = [Problem, {m, 1, M}, {n, 0, N}];
```



Generate the Block and boundary conditions in a single matrix

```
In[16]:= BlockMethod {};
In[17]:= Equations = Table[BlockMethod, {m, 1, M}, {n, 0, N-4, 4}] // Flatten;
In[18]:= Solution = Join[YVector, YpVector, YppVector, YpppVector];
In[19]:= MakeGuessPair[p_] := {p, 1}
InitialRootGuesses = Map[MakeGuessPair, Solution];
```

Solve Linear problem with NSolve

```
In[21]:= sol = FindRoot[Equations, InitialRootGuesses];
In[22]:= q0 = Table[u1,j /. sol, {j, 0, m}] // Flatten;
```

Output and numerical results

```
In[23]:= Exact[t_, x_] := (*exact solution if known*)
In[23]:= Vec = Table[Exact[t1, x[j]], {j, 0, m}] // Flatten;
In[24]:= Error = Abs[Vec - q0];
In[25]:= L^∞ = Max[Error];
In[26]:= L^1 = Norm1 = Norm[Vec - q0, 1];
In[27]:= L^2 = Norm2 = Norm[Vec - q0, 2];
In[28]:= RESULT = Table[{x[i-1], q0[[i]], Vec[[i]], q0[[i]] - Vec[[i]]}, {i, 1, m+1}];
In[29]:= Prepend[RESULT, {"x", "approx1", "exact1", "error1"}] // MatrixForm
```



Visual Representation of the solutions (3D-surface plot)

```
In[30]:= XTAX = Partition[XandT, 2];
In[31]:= XTUPoints = Partition[Flatten[Riffle[XTAX, q0G]], 3];
In[32]:= XTUExact = Partition[Flatten[Riffle[XTAX, UVec]], 3];
In[33]:= XTErrors = Partition[Flatten[Riffle[XTAX, Error]], 3];
In[34]:= ListPlot3D[XTUPoints, MaxPlotPoints -> 50, AxesLabel -> {"t", "x"},
PlotTheme -> "Detailed", PlotLabel -> "BUM-Numerical Solution"]

In[35]:= ListPlot3D[XTUExact, MaxPlotPoints -> 50, AxesLabel -> {"t", "x"},
PlotTheme -> "Detailed", PlotLabel -> "Exact Solution"]

In[36]:= ListPlot3D[XTErrors, MaxPlotPoints -> 50,
AxesLabel -> {"t", "x"}, PlotTheme -> "Detailed", PlotLabel -> "Error"]
```