

Subsemigroup Structure of the Rhotrix Bicyclic Semigroup

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Abstract

This paper investigates the subsemigroup structure of the rhotrix bicyclic semigroup obtained as a coordinatewise extension of the classical bicyclic semigroup. Order-defined subsemigroups determined by the componentwise relations $p_i \geq q_i$, $p_i \leq q_i$ and $p_i = q_i$ are introduced and analyzed. It is shown that the order-decreasing subsemigroup forms a subsemigroup but is neither regular nor inverse, with its regular elements coinciding precisely with the diagonal elements. The order-increasing subsemigroup is established to be anti-isomorphic to the order-decreasing case, while the diagonal subsemigroup is proved to be commutative idempotent semigroup and is isomorphic to a semilattice under componentwise maximum. These results provide structural classification of order-defined subsemigroups of the rhotrix bicyclic semigroup.

Keywords: Rhotrix bicyclic semigroup; order-defined subsemigroup; regularity; idempotent structure; inverse semigroup; semilattice.

INTRODUCTION

According to Howie[4], a semigroup is non-empty set equipped with a binary operation that is closed and associative. Among the fundamental examples in semigroup theory is the classical bicyclic semigroup, whose structure plays a significant role in the study of inverse and regular semigroups see Howie[5] and continues to inspire structural extensions.

The concept of rhotrix was introduced by Ajibade[2] as an extension of matrix-ternions and matrix-noitrets earlier proposed by Atanassov and Shannon[3]. Since then, rhotrices have been widely used as underlying sets for constructing algebraic systems. For example, Mohammed, Balarabe and Imam[12] constructed semigroups based on rhotrix sets together with the rhotrix multiplication introduced by Sani[17]. Ahmed, Yisa and Ayinla[1] investigated orthogonal rhotrix groups and characterized their principal subgroups, while Mohammed and Okon[12] studied non-commutative rhotrix groups and established subgroup embedding into the classical general linear group. Further developments by Isere[6] introduced even-dimensional rhotrices and generalized multiplication for higher even dimensions, extending earlier work that focused on odd dimensions.

In our recently accepted paper, “The Concept of Rhotrix Bicyclic Semigroup”, we introduced the rhotrix bicyclic semigroup as a coordinatewise extension of the classical bicyclic semigroup and established its basic algebraic properties. Motivated in part by structural investigations of certain subgroups of the bicyclic semigroup by Makanjuola and Umar[8], the present work continues their study by examining internal subgroups defined through coordinatewise order relations.

Specifically, we analyze subsets determined by the conditions $p_i \geq q_i, p_i \leq q_i$ and $p_i = q_i$. We establish closure and associativity, investigate regularity and inverse properties, describe the idempotent structure, and determine the relationship between the order-decreasing and order-increasing subsemigroups. In particular, the diagonal subsemigroup is shown to form a commutative idempotent semigroup with a semilattice structure under componentwise maximum.

These results further clarify the internal structure of the rhotrix bicyclic semigroup and reinforce the role of rhotrices as a flexible foundation for extending classical semigroup constructions.

Preliminary

The definitions below lay the groundwork for the subsequent sections.

Definition 2.1 (Rhotrix)

A rhotrix is a rhomboidal arrangement of real numbers which are some ways, between (2×2) dimensional and (3×3) dimensional matrices of the form

$$P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in R$$

Where c is the heart of the rhotrix P denoted as $h(P)$.

Definition 2.2 (Addition of rhotrices)

The sum of two rhotrices P and Q is defined as follows:

$$P + Q = \left\langle \begin{array}{ccc} a & & \\ b & h(P) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} f & & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} a + f & & \\ b + g & h(P) + h(Q) & d + j \\ & e + k & \end{array} \right\rangle$$

Definition 2.3 (Multiplication of Rhotrices)

The multiplication of heart based rhotrices was first introduced by Ajibade[] and is defined as follows:

$$P \circ Q = \left\langle \begin{array}{ccc} & ah(Q) + fh(P) & \\ bh(Q) + gh(P) & h(P)h(Q) & dh(Q) + jh(P) \\ & eh(Q) + kh(P) & \end{array} \right\rangle$$

An alternative multiplication method for heart-based rhotrices, known as the “row-column” method, was introduced by Sani[16] and is defined as follows:

$$P \circ Q = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & h(P)h(Q) & aj + dk \\ & bj + ek & \end{array} \right\rangle$$

We now recall the classical bicyclic structure and its coordinatewise extension to the rhotrix setting.

Definition 2.4 (Classical Bicyclic Semigroup)

Let $N_0 := 0, 1, 2, \dots$

The classical bicyclic semigroup is defined as

$$B := \{(m, n) : m, n \in N_0\},$$

With binary operation $\circ$ given by

$$(m, n) \circ (r, s) = (m + r - \min(n, r), n + s - \min(n, r))$$

It is well known that B is associative and forms a fundamental example in semigroup theory.

Definition 2.5 (Rhotrix Bicyclic Set)

Let $k \in N_0$ be fixed

For $P = (p_1, p_2, \dots, p_k)$ and $Q = (q_1, q_2, \dots, q_k)$ in N_0^k , define

$$B(N_0) := \{(P, Q) : P, Q \in N_0^k\}.$$

Multiplication on $B(N_0)$ is defined componentwise by

$$(P, Q) * (R, S) = ((p_i + r_i - \min(q_i, r_i), (q_i + s_i - \min(q_i, r_i)))_{i=1}^k$$

Thus, multiplication in each coordinate follows the classical bicyclic rule. Consequently, associativity of $B(N_0)$ follows from associativity in each coordinate.

Note that rhotrices are denoted by capital letters, while their entries are denoted componentwise by lower case letters.

Definition 2.6 (Order-Defined Subsemigroups)

We now introduce subsemigroups determined by coordinatewise order relations.

Order-decreasing subsemigroup

$$B^-(N_0) = \{(P, Q) \in B(N_0) : p_i \geq q_i \text{ for all } i\}.$$

Order-increasing subsemigroup

$$B^+(N_0) = \{(P, Q) \in B(N_0) : p_i \leq q_i \text{ for all } i\}$$

Diagonal(order-equality) subsemigroup

$$B^0(N_0) = \{(P, Q) \in B(N_0) : p_i = q_i \text{ for all } i\}$$

It is immediate that

$$B^0(N_0) = B^-(N_0) \cap B^+(N_0)$$

We now proceed to investigate the structural properties of these order-defined subsemigroups, beginning with $B^-(N_0)$.

Subsemigroup Structure of $B^-(N_0)$

We begin by establishing that $B^-(N_0)$ forms a subsemigroup of the rhotrix bicyclic semigroup $B(N_0)$ which is the coordinatewise extension of the classical bicyclic semigroup.

Theorem 3.1

$B^-(N_0)$ is closed under the operation of $B(N_0)$.

Proof

Scalar Case:

Let (m, n) and (r, s) be elements of the classical bicyclic semigroup satisfying $m \geq n$ and $r \geq s$.

Their product

$$(m, n) \circ (r, s) = (m + r - \min(n, r), n + s - \min(n, r)).$$

$$\text{Let } x = m + r - \min(n, r), \quad y = n + s - \min(n, r)$$

We show that $x \geq y$

$$\text{Observe that } x - y = (m - n) + (r - s).$$

Since $m \geq n$ and $r \geq s$, both terms on the right-hand side are non-negative. Hence, $x - y \geq 0$, so $x \geq y$.

Thus, the order condition is preserved in the scalar case.

Componentwise Case:

Let $(P, Q), (R, S) \in B^-(N_0)$. Then for every coordinate i , $p_i \geq q_i$ and $r_i \geq s_i$. By scalar argument above, $p_i + r_i - \min(q_i, r_i) \geq q_i + s_i - \min(q_i, r_i)$.

Hence, the product $(P, Q) * (R, S)$ satisfies the order condition in every coordinate. Therefore, $(P, Q) * (R, S) \in B^-(N_0)$.

Thus, $B^-(N_0)$ is closed under the binary operation $*$.

Theorem 3.2

The operation $*$ on $B^-(N_0)$ is associative.

Proof

Since $B(N_0)$ is defined componentwise from the classical bicyclic semigroup, and the classical bicyclic semigroup is associative, it follows that multiplication in each coordinate is associative.

Therefore, for all

$$(P, Q), (R, S), (T, U) \in B^-(N_0),$$

$$((P, Q) * (R, S)) * (T, U) = (P, Q) * ((R, S) * (T, U))$$

Hence, $B^-(N_0)$ is associative.

Corollary 3.1

$(B^-(N_0), *)$ is a subsemigroup of $(B(N_0), *)$.

Example 3.1

$$\left[\left(\begin{array}{c} \langle 3 \\ 2 \ 3 \ 4 \\ 5 \end{array} \right), \left(\begin{array}{c} \langle 2 \\ 2 \ 2 \ 3 \\ 4 \end{array} \right) * \left(\begin{array}{c} \langle 4 \\ 3 \ 2 \ 1 \\ 4 \end{array} \right), \left(\begin{array}{c} \langle 3 \\ 2 \ 2 \ 1 \\ 3 \end{array} \right) \right] * \left[\left(\begin{array}{c} \langle 1 \\ 3 \ 2 \ 4 \\ 3 \end{array} \right), \left(\begin{array}{c} \langle 1 \\ 3 \ 2 \ 3 \\ 1 \end{array} \right) \right]$$

$$\begin{aligned}
 &= \left(\begin{array}{c} \left(\begin{array}{ccc} & 5 & \\ 3 & 3 & 4 \\ & 5 & \end{array} \right), \left(\begin{array}{ccc} & 3 & \\ 2 & 2 & 3 \\ & 3 & \end{array} \right) * \left(\begin{array}{ccc} & 1 & \\ 3 & 2 & 4 \\ & 3 & \end{array} \right), \left(\begin{array}{ccc} & 1 & \\ 3 & 2 & 3 \\ & 1 & \end{array} \right) \end{array} \right) \\
 &= \left(\begin{array}{c} \left(\begin{array}{ccc} & 5 & \\ 4 & 3 & 5 \\ & 5 & \end{array} \right), \left(\begin{array}{ccc} & 3 & \\ 3 & 2 & 3 \\ & 1 & \end{array} \right) \end{array} \right)
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 &\left(\begin{array}{c} \left(\begin{array}{ccc} & 3 & \\ 2 & 3 & 4 \\ & 5 & \end{array} \right), \left(\begin{array}{ccc} & 2 & \\ 2 & 2 & 3 \\ & 4 & \end{array} \right) * \left[\left(\begin{array}{ccc} & 4 & \\ 3 & 2 & 1 \\ & 4 & \end{array} \right), \left(\begin{array}{ccc} & 3 & \\ 2 & 2 & 1 \\ & 3 & \end{array} \right) * \left(\begin{array}{ccc} & 1 & \\ 3 & 2 & 4 \\ & 3 & \end{array} \right), \left(\begin{array}{ccc} & 1 & \\ 3 & 2 & 3 \\ & 1 & \end{array} \right) \right] \end{array} \right) \\
 &= \left(\begin{array}{c} \left(\begin{array}{ccc} & 3 & \\ 2 & 3 & 4 \\ & 5 & \end{array} \right), \left(\begin{array}{ccc} & 2 & \\ 2 & 2 & 3 \\ & 4 & \end{array} \right) * \left(\begin{array}{ccc} & 4 & \\ 4 & 2 & 4 \\ & 4 & \end{array} \right), \left(\begin{array}{ccc} & 3 & \\ 3 & 2 & 3 \\ & 1 & \end{array} \right) \end{array} \right) \\
 &= \left(\begin{array}{c} \left(\begin{array}{ccc} & 5 & \\ 4 & 3 & 5 \\ & 5 & \end{array} \right), \left(\begin{array}{ccc} & 3 & \\ 3 & 2 & 3 \\ & 1 & \end{array} \right) \end{array} \right)
 \end{aligned}$$

Theorem 3.3

The scalar order-decreasing subsemigroup of B^- is not regular.

Proof

Let $a = (m, n) \in B^-$ with $m > n$. In the full bicyclic semigroup, the unique inverse of (m, n) is

$$(m, n)^{-1} = (n, m).$$

Regularity requires the existence of $x \in B^-$ such that $a = axa$.

From bicyclic structure theory, any element satisfying $a = axa$ must be the inverse (n, m) .

However,

$$(n, m) \in B^- \Leftrightarrow n \geq m,$$

which contradicts $m > n$. Hence no such $x \in B^-$ exists. Therefore, a is not regular in B^- . Since there exists at least one regular element, B^- is not regular.

Proposition 3.1 (Characterization of Regular Elements)

An element $(m, n) \in B^-$ is regular if and only if $m = n$.

Proof

If $m = n$, then $(m, m)^\circ(m, m)^\circ(m, m) = (m, m)$, so the element is regular. Conversely, If $m > n$, from Theorem 3.3 the element is not regular.

Now let us pass to the rhotrix structure.

Let

$$B^-(N_0) = \{(P, Q) \in B(N_0) : p_i \geq q_i \text{ for all } i\},$$

where multiplication is defined componenewise from the scalar bicyclic operation.

Theorem 3.4

The rhotrix order-decreasing bicyclic subsemigroup $B^-(N_0)$ is not regular.

Proof

Let $A = (P, Q) \in B^-(N_0)$, so that $p_i \geq q_i$ for every component. Regularity would require the existence of $X = (R, S) \in B^-(N_0)$ such that $A = AXA$.

Since multiplication is defined componenewise, this equality holds if and only if for each i ,

$$(p_i, q_i) = (p_i, q_i)(r_i, s_i)(p_i, q_i)$$

in the scalar bicyclic semigroup. But from Proposition 3.1, this is possible only when $p_i = q_i$ for each i . Hence, if there exists at least one index i with $p_i > q_i$, then A is not regular. Therefore, $B^-(N_0)$ is not regular.

Corollary 3.2

An element $(P, Q) \in B^-(N_0)$ is regular if and only if $P = Q$ componenewise.

Idempotent Structure of $B^-(N_0)$

Theorem 3.5

The idempotents of the bicyclic semigroup B are precisely the elements of the form (n, n) , $n \in N_0$.

Proof

Let $(m, n) \in B$ be idempotent. Then $(m, n)^2 = (m, n)$.

Compute:

$$(m, n)(m, n) = (m + m - \min(n, m), n + n - \min(n, m))$$

Let $k = \min(n, m)$. Then, $(m, n)^2 = (2m - k, 2n - k)$.

For idempotency, we require $(2m - k, 2n - k) = (m, n)$. Thus, $2m - k = m$ and $2n - k = n$.

Hence, $m = k$ and $n = k$. So $m = n$.

Conversely, if $m = n$, then $(n, n)^2 = (n, n)$. Thus, the idempotents are exactly (n, n) .

Idempotents in the Scalar Order-Decreasing Subsemigroup:

Recall

$$B^- := \{(m, n) \in B : m \geq n\}.$$

Since the scalar idempotents are (n, n) , and these satisfy $n \geq n$, we obtain:

Corollary 3.3

The idempotents of B^- are precisely

$$E(B^-) = \{(n, n) : n \in N_0\}$$

Thus, the order restriction does not change the idempotent set.

Proposition 3.2 (Commutativity of Idempotents)

For $m, n \in N_0$

$$(m, m)^\circ(n, n) = (m + n - \min(m, n), m + n - \min(m, n)).$$

This simplifies to $(\max(m, n), \max(m, n))$, since the expression is symmetric in m, n , we have

$$(m, m)^\circ(n, n) = (n, n)^\circ(m, m)$$

Hence the idempotents commute.

Idempotents in the Rhotrix Order-Decreasing Subsemigroup

Now consider the rhotrix order-decreasing subsemigroup

$$B^-(N_0) := \{(P, Q) : p_i \geq q_i \text{ for all } i\}.$$

Multiplication is defined componentwise.

Theorem 3.6

The idempotents of $B^-(N_0)$ are precisely the elements of the form: (P, P) .

Proof

Let $A = (P, Q) \in B^-(N_0)$ be idempotent. Then $A^2 = A$. Since multiplication is componentwise,

This holds if and only if

$$(p_i, q_i)^2 = (p_i, q_i) \text{ for all } i$$

From the scalar result, this implies $p_i = q_i$ for all i . Hence $P = Q$.

Conversely, if $P = Q$, then $(P, P)^2 = (P, P)$.

Thus, idempotents are exactly those of the form (P, P) .

Inverse Property of the Order-Decreasing Subsemigroup

Theorem 3.7

The scalar order decreasing subsemigroup B^- is not an inverse semigroup.

Proof

Let $a = (m, n) \in B^-$ with $m > n$.

If B^- were inverse, there would exist $b \in B^-$ such that $a = aba$ and $b = bab$. But in the bicyclic semigroup, the inverse of (m, n) is uniquely (n, m) . Since $(n, m) \notin B^-$, no such b exists in B^- . Hence B^- is not inverse.

Proposition 3.3

An element $(m, n) \in B^-$ possesses an inverse in B^- if and only if $m = n$.

Proof

If $m = n$, then $(m, m)^{-1} = (m, m)$, and clearly $(m, m) \in B^-$.

Conversely, if $m > n$ its inverse lies outside B^- .

Now consider the rhotrix order-decreasing subsemigroup

$$B^-(N_0) = \{(P, Q) \in B(N_0) : p_i \geq q_i \text{ for all } i\}.$$

Multiplication is defined componentwise.

Theorem 3.8

The rhotrix order-decreasing bicyclic subsemigroup $B^-(N_0)$ is not an inverse semigroup.

Proof

Let $A = (P, Q) \in B^-(N_0)$ with $p_j = q_j$ for some index j . In full rhotrix bicyclic semigroup, the inverse is $A^{-1} = (Q, P)$.

But $(Q, P) \in B^-(N_0) \Leftrightarrow q_i \geq p_i$ for all i , which contradicts $p_j > q_j$. Hence, A does not possess an inverse in $B^-(N_0)$. Therefore, $B^-(N_0)$ is not inverse.

Corollary 3.4

An element $(P, Q) \in B^-(N_0)$ has an inverse in $B^-(N_0)$ if and only if $P = Q$ componenewise.

Commutative of the Order-Decreasing Subsemigroup

Theorem 3.9

The scalar order-decreasing subsemigroup B^- is not commutative.

Proof

To prove non-commutativity, it suffices to prove a counter example.

Take $a = (2, 1)$, $b = (1, 0)$. Both satisfy $m \geq n$, hence belong to B^- .

Compute $ab: (2, 1)(1, 0) = (2 + 1 - \min(1, 1), 1 + 0 - \min(1, 1)) = (2, 0)$, since $\min(1, 1) = 1$

Compute $ba: (1, 0)(2, 1) = (1 + 2 - \min(0, 2), 0 + 1 - \min(0, 2)) = (3, 1)$, since $\min(0, 2) = 0$

Clearly $ab \neq ba$.

Hence B^- is not commutative.

Theorem 3.10

The rhotrix order-decreasing bicyclic subsemigroup $B^-(N_0)$ is not commutative.

Proof

Since multiplication is defined componenewise, commutativity in $B^-(N_0)$ would imply commutativity in scalar coordinate. But from Theorem 3.9, the scalar subsemigroup B^- is not commutative. Hence, there exist scalar components a_i, b_i such that $a_i b_i \neq b_i a_i$.

Construct rhotrix elements whose entries agree with this counter example in one coordinate and are equal elsewhere. Then the resulting rhotrix products differ in that coordinate. Therefore, $AB \neq BA$. Hence $B^-(N_0)$ is not commutative.

The Order-Increasing Rhotrix Bicyclic Subsemigroup

We now show that the order-increasing structure is completely determined by the order-decreasing one via an anti-isomorphism. Consequently, no repetition of proofs is required.

Scalar Anti-Isomorphism

Let $B := \{(m, n) : m, n \in N_0\}$ be the classical bicyclic semigroup.

Define map $\delta: B \rightarrow B$ by $\delta(m, n) = (n, m)$.

Proposition 4.1

The mapping δ is an anti-isomorphism of B .

Proof

Let $(m, n), (r, s) \in B$. Recall the multiplication:

$(m, n) \circ (r, s) = (m + r - \min(n, r), n + s - \min(n, r))$. Then

$\delta((m, n) \circ (r, s)) = (n + s - \min(n, r), m + r - \min(n, r))$.

On the other hand,

$$\delta(r, s)\delta(m, n) = (s, r)(n, m)$$

Compute: $(s, r) \circ (n, m) = (s + r - \min(r, n), r + m - \min(r, n))$. Since $\min(n, r) = \min(r, n)$, both expressions coincide. Hence,

$$\delta((m, n) \circ (r, s)) = \delta(r, s)\delta(m, n).$$

Thus δ reverses multiplication order and being clearly bijective, it is an anti-isomorphism.

Between B^- and B^+

Define: $B^- := \{(m, n) \in B : m \geq n\}$ and $B^+ := \{(m, n) \in B : m \leq n\}$

Theorem 4.2

$$\delta(B^-) = B^+$$

Hence B^- and B^+ are anti-isomorphic.

Proof

Let $(m, n) \in B^-$. Then $m \geq n$. Thus, $\delta(m, n) = (n, m)$, and since $n \leq m$, we have $(n, m) \in B^+$.

Conversely, if $(a, b) \in B^+$, then $a \leq b$, so $(b, a) \in B^-$, and $\delta(b, a) = (a, b)$. Thus, $\delta(B^+) = B^-$.

Since δ is an anti-isomorphism of B , its restriction gives an anti-isomorphism between B^- and B^+ .

Now define the rhotrix order-increasing subsemigroup

$$B^+(N_0) := \{(P, Q) \in B(N_0) : p_i \leq q_i \text{ for all } i\}$$

Define $\beta : B(N_0) \rightarrow B(N_0)$ by $\beta(P, Q) = (Q, P)$

Theorem 4.3

The mapping β is an anti-isomorphism of the rhotrix bicyclic semigroup, and $\beta(B^-(N_0)) = B^+(N_0)$.

Since multiplication is defined componentwise, for any $(P, Q), (R, S)$,

$$\beta((P, Q), (R, S)) = \beta(R, S)\beta(P, Q) \text{ componenetwise by Proposition 4.1.}$$

Moreover,

$$(P, Q) \in B^-(N_0) \Leftrightarrow p_i \geq q_i \text{ for all } i,$$

$$\Leftrightarrow q_i \leq p_i \text{ for all } i, \Leftrightarrow (Q, P) \in B^+(N_0).$$

The order-increasing rhotrix bicyclic subsemigroup $B^+(N_0)$ is anti-isomorphic to the order-decreasing subsemigroup $B^-(N_0)$.

Therefore, all structural properties established for $B^-(N_0)$ holds dually for $B^+(N_0)$.

Order-Equality Rhotrix Bicyclic Semigroup

We now analyse the order-defined subsemigroup

$$B^0(N_0) := \{(P, Q) \in B(N_0) : p_i = q_i \text{ for all } i\}.$$

Scalar Diagonal Subsemigroup

Let $B := \{(m, n) : m, n \in N_0\}$ be the classical bicyclic semigroup. Define the diagonal subset

$$B^0 := \{(n, n) : n \in N_0\}.$$

Theorem 5.1

B^0 is a commutative subsemigroup of B .

Proof

Take $(m, m), (n, n) \in B^0$

Compute: $(m, m) \circ (n, n) = (m + n - \min(m, n), m + n - \min(m, n))$

$$\text{Since } m + n - \min(m, n) = \max(m, n)$$

$(m, m)(n, n) = (\max(m, n), \max(m, n))$. Thus, the product remains diagonal, so the set is closed.

Because $\max(m, n) = \max(n, m)$,

$(m, m)(n, n) = (n, n)(m, m)$. Hence, it is commutative.

The structural interpretation of this result is that the operation reduces to:

$(m, m)(n, n) = (\max(m, n), \max(m, n))$. Thus, B^0 is isomorphic to $(N_0, \max)$.

Theorem 5.2

$B^0(N_0)$ is a commutative idempotent subsemigroup of $B(N_0)$.

Proof

Let $(P, P), (R, R) \in B^0(N_0)$. Since multiplication is componentwise,

$(P, P) * (R, R) = (\max(p_i, r_i), \max(p_i, r_i))$. Thus:

- (i) The product is diagonal
- (ii) Every element satisfies $(P, P)^2 = (P, P)$

Therefore, $B^0(N_0)$ is a commutative band (indeed semilattice).

$$B^0(N_0) = B^-(N_0) \cap B^+(N_0)$$

So, the diagonal is the intersection of the order-decreasing and order-increasing subsemigroups.

Conclusion

This paper extends our earlier work on the rhotrix bicyclic semigroup by examining internal subsemigroups defined through coordinatewise order relations. We showed that the order-decreasing subset forms subsemigroup but is neither regular nor

inverse in general, and that its regular elements coincide with the diagonal elements. The order-increasing subsemigroup was shown to possess corresponding structural properties, while the diagonal subsemigroup was identified as a commutative idempotent semigroup admitting a semilattice structure under componentwise maximum.

These findings provide clearer structural decomposition of the rhotrix bicyclic semigroup and further illustrate how order constraints shape the algebraic behavior of coordinatewise extensions of the classical bicyclic semigroup.

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